Simple item auction price of anarchy

n players $v_1 \ldots v_n$ value for item

SW giving item to bidder i at price p

$$SW = \underbrace{(v_i - p)}_{\text{value for winner}} + \underbrace{p}_{\text{value for auctioneer}} + \underbrace{0 \cdots + 0}_{\text{value for others}} = v_i$$

Way to prove PoA:

$u_i(s_i^*, s_{-i})$

s_i^* special strategy

s any other strategy vector

Example: bid $v_i/2$

Claim s any bid-vector

$$u_i\left(\frac{v_i}{2}, s_{-i}\right) \geq \frac{1}{2} v_i - \text{price}(s)$$

$$\text{price}(s) = \max_j s_j$$

Proof: • OK if $\frac{v_i}{2}$ wins

$$\Rightarrow u_i\left(\frac{v_i}{2}, s_{-i}\right) = v_i - \frac{v_i}{2}$$

• if $v_i/2$ looses

$$\text{price}(s) > \frac{v_i}{2}$$

also true $\boxed{u_i(v_i/2, s_{-i}) \geq 0}$

Price of anarchy proof

$$\boxed{\sum_i u_i\left(\frac{v_i}{2}, s_{-i}\right) \geq \frac{1}{2} \max_j v_j - \text{price}(s)}$$

reasons use top for argmax v_j

second one for the rest

Assume s Nash (pure strategy)

$$\underbrace{\sum_i u_i(s)}_{} \geq \sum_i u_i\left(\frac{v_i}{2}, s_{-i}\right) \geq \underbrace{\frac{1}{2} \max_j v_j}_{} \underbrace{- \text{price}(s)}_{}$$

$$SW(s) \geq \frac{1}{2} \max_j v_j = \frac{1}{2} OPT$$

general framework for PoA in auctions

$$\sum_i u_i(s_i^*, s_{-i}) \geq \lambda\, \mathrm{OPT} - \mathrm{Rev}(s)$$

opt social welfare

revenue for equilibrium bids

$\Rightarrow$ $SW(s) \geq \lambda \cdot \mathrm{OPT}$

all Nash equilibria s

Next mixed Nash: σ Nash

same true: use expectations

$$\sum_i E_{s\sim\sigma}(u_i(s)) \geq \sum_i E_{s\sim\sigma}\left(u_i\left(\tfrac{v_i}{2}, s_{-i}\right)\right)$$

$$\geq \cancel{E_{s\sim\sigma}}(\lambda \cdot \mathrm{OPT}) - E_{s\sim\sigma}(\mathrm{price}(A))$$

& no-regret learning outcome

$$\sum_t u_i(s^t) \geq \sum_t u_i\left(\tfrac{v_i}{2}, s_{-i}^t\right) - \mathrm{Reg}$$

discretize options εv_i rounding

$\frac{1}{\varepsilon}$ options to bid $\mathrm{Reg} = \sqrt{2 \ln \frac{1}{\varepsilon} T}$

$$\boxed{\sum_t \sum_i u_i(s^t)} \geq \sum_t \sum_i u_i\left(\frac{v_i}{2}, s_{-i}^t\right) - n\,Reg$$

$$\geq \boxed{\sum_t \frac{1}{2} \max_j v_j} - \boxed{\sum_t price(s^t)} - n\,Reg$$

$$\sum_t SW(s^t) \geq \frac{1}{2} T \max_j v_j - n\,Reg$$

All pay! need

$$\sum_i u_i(s_i^*, s_{-i}) \geq \lambda \max_j v_j - Rev(s)$$

j not max value $s_j^* = 0$

$$u_i(0, s_{-i}) \geq 0$$

j is max s_j^* unif $[0, v_j]$

$$E_{s_j^*}\left(u_j(s_j^*, s_{-j})\right)$$

$$= \frac{1}{v_j} \int_0^{v_j} u_j(x, s_{-j})\,dx$$

$$= v_j\left(\frac{v_j - p}{v_j}\right) - \frac{v_j}{2} = \frac{1}{2} v_j - p$$

exp payment

$p = \max_{i \neq j} s_i$

prob of winning $\frac{v_j - p}{v_j}$

Result:

$$E\left(\sum_i u_i(s_i^*, s_{-i})\right) \geq \frac{1}{2} \max_j v_j - Rev(s)$$

$$\implies SW(s) \geq \frac{1}{2} OPT$$