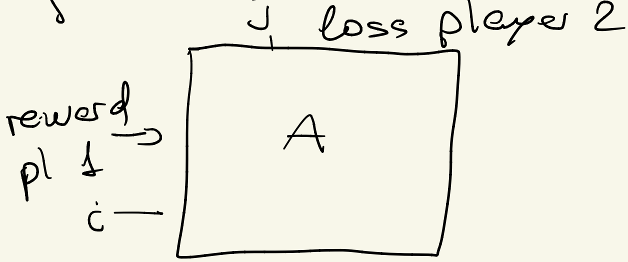
2-person 0 sum

two no-regret learners



entries $[-1, 1]$

$x^1 \dots x^t$ prob dist player 1

$y^1 \dots y^t$ —"— player 2

$$\ell^c_t(j) = \sum_i a_{ij} x_i^t$$ expected loss

$$\ell^r_t(i) = -\sum_j a_{ij} y_j^t$$ —"—

Expected payoff player 1 $\sum_t (x^t)^T A y^t$

no-regret player 2:

$$(*) \quad \sum_t (x^t)^T A y^t \le \min_j \sum_t \sum_i a_{ij} x_i^t$$

player 1:

$$(*) \quad -\sum_t (x^t)^T A y^t \le \min_i -\sum_t \sum_j a_{ij} y_j^t$$

$\bar{x} = \frac{1}{T}\sum_t x^t$, $\bar{y} = \frac{1}{T}\sum_t y^t$

Claim $\bar{x}$ & $\bar{y}$ is Nash & v = value of the game

T = time played

$$v = \frac{1}{T}\sum (x^t)^T A y^t$$

Player 2:

(**) $v \cdot \cancel{T} \le \min_j \cancel{T} \sum_i a_{ij}\bar{x}_i$

if player 1 commits to play $\bar{x}$ then

guaranteed to gain $\ge v$

if player 2 commits to play $\bar{y}$ then

(**) $v \cdot \cancel{T} \ge \cancel{T} \max_i \sum_j a_{ij}\bar{y}_j$

guaranteed to loose $\le v$

$\Longrightarrow$ $\bar{x}$ & $\bar{y}$ Nash & v is the value

Proof:

(1) $\bar{x}^T A \bar{y} \le v$ by second line

$\ge v$ by first row

(2) two rows show no deviation.

Adding error: (in existance proof)

difference in value

~~colum~~ value for player 1

Δ = value if row chooses first

− value if column chooses first

Recall Regret bound from multiplicative weight alg.

$\sqrt{T 2 \ln k}$ $\quad k = \#$ strategies

added term in (∗)

top ∗∗ get added term $\sqrt{\frac{2 \ln k}{T}}$

bottom ∗∗ get $\sqrt{\frac{2 \ln k}{T}}$ subtracted

choose T so that $\sqrt{\frac{2 \ln k}{T}} < \frac{\Delta}{2}$

$\Rightarrow$ if row commits to $\bar{x}$

guarated to gain $\geq v - \frac{\Delta}{2}$

if column commits to $\bar{y}$

- guaranteed to loose $< v + \frac{\Delta}{2}$

$\Rightarrow$ going first difference ~~sta~~

strictly less than

$$v + \frac{\Delta}{2} - \left(v - \frac{\Delta}{2}\right) = \Delta$$

contradiction.

Alternate version

loss defined by random of opponent

$i_1 \ldots i_t \qquad j_1 \ldots j_t$

~~tos~~ value for player 1

$$\sum_t a_{i_t j_t}$$

$$\ell_t^c(j) = a_{i_t j} \; ; \quad \ell_t^r(i) = -a_{i j_t}$$

compute expectation

$$E\left(\sum_{t=1}^{T} a_{i_t j_t}\right) = \sum_{t=1}^{T} (x^t)^T A y^t$$

$$E\left(\sum_{t=1}^{T} a_{i_t j}\right) = \sum_t \sum_i x_i^t a_{ij}$$