Algorithmic collusion

same product sold by 2 competitors

prices $\in \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k}{k} = 1$

volume 1 of buyers interested in product

all buy cheaper option

or $\frac{1}{2} - \frac{1}{2}$ if price equal

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Monopoly pricing: goal seller: max revenue

$p = 1 \Rightarrow$ revenue $= \frac{1}{2}$ each

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Nash equilibrium:

$\left(\frac{1}{k}, \frac{1}{k}\right), \left(\frac{2}{k}, \frac{2}{k}\right)$

Claim all Nash prices are $\leq \frac{2}{k}$

by induction: $p$ max price offered

only player does $p$

$\Rightarrow$ wins nothing at $p$

both do $p$ with prob $a, b$ resp.

~~value~~ revenue for a at price $p$

$b \cdot \frac{1}{2} \cdot p$

other price same

revenue change do this replacing $p$ with $p - 1/k$

$$\geq b \; 1 \left(p - \frac{1}{k}\right)$$

if $p > \frac{2}{k} \implies$ new revenue better.

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Assume opponent is no-swap regret learner

Opponent (optimizer) need Steckelberg strategy

One proposal: use uniformly ~~p~~ random price

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Question: what is best response?

revenue of price $p$: for the learner

| $p$ | | prob | revenue (if winning) |
|---|---|---|---|
| opponent | $> p$ | $1-p$ | $1 \cdot p$ |
| " | $= p$ | $1/k$ | $\frac{1}{2} p$ |
| " | $< 0$ | | $0$ |

total expected revenue

$$(1-p)\, 1 \cdot p + \frac{1}{k} \cdot \frac{1}{2} p$$

& calculus opt $= \frac{2k+1}{4k} \approx \frac{1}{2}, \; \frac{1}{2} + \frac{1}{k}$

assuming $k$ even

best bid allowed $\frac{1}{2}$

$\Longrightarrow$ revenue $\frac{1}{4} + \frac{1}{4k} > \frac{1}{k}$ ↖ Nash

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Given this best response

Optimizer's revenue:

| | | | |
|---|---|---|---|
| $p$ | $p > \frac{1}{2}$ | | $0$ |
| | $\frac{1}{2}$ | $\frac{1}{k}$ | $\frac{1}{2} \cdot \frac{1}{2}$ |
| | $p < \frac{1}{2}$ | | |

$$\frac{1}{k} \sum_{i=1}^{\frac{k}{2}-1} \frac{i}{k} = \frac{1}{k} \frac{\frac{k}{2}\left(\frac{k}{2}-1\right)}{2k} = \frac{1}{8} - \frac{1}{4k}$$

$p = i/k$

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<u>Claim</u> if leader has revenue $\geq c$ const indep of $k$

$\Longrightarrow$ follower is also $\geq$ constant

Proof: leader must use price $> c/2$ at least prob $c/2$

One option for learner $p = \frac{c}{2} - \frac{1}{k}$

revenue $\geq \frac{c^2}{4} - \frac{c}{k}$

due to winning when leader $> c/2$