Follow the perturbed leader

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time t ect

$$s^* = \arg\max_s \sum_{\tau=1}^{t-1} u_i(s, s_{-i}^\tau)$$

our framework:

adversary sets $u^t(s)$ $s \in S$

$$s^t = \arg\max_s \sum_{\tau=1}^{t-1} u^\tau(s)$$

Robinson '56 : 2-person 0-sum games

follow the leader converges to Nash

in marginal dist

Cheat algorithm: be the leader

$$s^t = \arg\max \sum_{\tau=1}^{t} u^\tau(s)$$

Claim: sequence has no-regret:

player has strategy set S

adversary sets $u^t : S \to \mathbb{R}$

Proof: induction

$t=1$ by choice

$t>1$

$$\sum_{\tau=1}^{t} u^\tau(s^\tau) = \sum_{\tau=1}^{t-1} u^\tau(s^\tau) + u^t(s^t)$$

$$\overset{\text{i.h}}{\geq} \max_s \sum_{\tau=1}^{t-1} u^\tau(s) + u^t(s^t)$$

$$\geq \sum_{\tau=1}^{t} u^\tau(s^t) \geq \max_s \sum_{\tau=1}^{t} u^\tau(s)$$

by choice of s^t

Add noise:

$$s^t = \arg\max_s \left(\sum_{\tau=1}^{t} u^\tau(s) + \xi_s \right)$$

indep random

Claim: regret of this sequence in exp

$$\leq E(\max_s \xi_s)$$

Proof: base $t=0$ ✓ ✓

$$E\Big(\sum_{\tau=1}^{t} u^\tau(s^\tau)\Big) = E\Big(\sum_{\tau=1}^{t-1} u^\tau(s^\tau)\Big) + E(u^t(s^t))$$

$$\overset{\text{i.h.}}{\geq} \max_s E\Big(\sum_{\tau=1}^{t-1} u^\tau(s) \cancel{- E(\max_s \xi_s)}\ + \xi_s\Big)$$

$$+ E(u^t(s^t))$$

$$\geq E\Big(\sum_{\tau=1}^{t} u^\tau(s^t)\Big) + \xi_{s^t} - E(\max_s \xi_s)$$

$$\geq E\Big(\sum_{\tau=1}^{t} u^\tau(s) + \xi_s - \cancel{\xi_{s^t}}\Big) - E(\max_s \xi_s)$$

stronger i.h.

$$E\Big(\sum_{\tau=1}^{t} u^\tau(s^\tau)\Big) + E(\max_s \xi_s) \geq E\Big(\sum_{\tau=1}^{t} u^\tau(s) + \xi_s\Big)$$

for all s

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implies desired statement if $\xi_s \geq 0$

Next step:

1. define ξ_s need $E(\max_s \xi_s)$

2. Prob $\arg\max \sum_{\tau=1}^{t} u^\tau(s) + \xi_s$

$$= \arg\max \sum_{\tau=1}^{t-1} u^\tau(s) + \xi_s$$

$\geq 1 - \varepsilon$

$\Pr(P > Q+1) > 1-\varepsilon$

$\boxed{(P > Q+1) \le (1-\varepsilon)^{P\,|\,Q}}$

all 3 changed: $(1-\varepsilon)^3$ ✓

≥ 2 changed $(1-\varepsilon)^2$

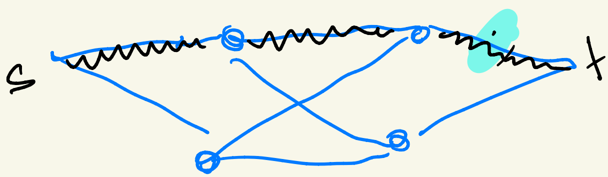
not bigger & only changed ✗

$(1-\varepsilon)^k$

~~do~~ repeat: $\boxed{\frac{k \ln E}{\varepsilon}}$

$(1-\varepsilon)^k \sim \boxed{1-k\varepsilon}$ only 1 bigger $(1-\varepsilon)$

all same



Last time

$$s^t = \arg\max_s \sum_{\tau=1}^{t} u^t(s)$$

$\Longrightarrow$ no-regret

$$\sum_{\tau=1}^{t} u^{\tau}(s^{\tau}) \geq \max_s \sum_{\tau=1}^{t} u^{\tau}(s)$$

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add arbitrary $\xi_s \geq 0$ all s

<u>Claim</u> using $s^t = \arg\max \sum_{\tau=1}^{t} u^{\tau}(s) + \xi_s$

$$\sum_{\tau=1}^{t} u^{\tau}(s^{\tau}) + \max_s \xi_s \geq \max_s \sum_{\tau=1}^{t} u^{\tau}(s) + \xi_s$$

proof: same as above
with step 0 $u^0(s) = \xi_s$

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real follow perturbed leader
ξ_s & used s^{t-1} in step t

ξ_s choosen random: $Pr(s^{t-1} = s^t)$

ξ_s coin H = prob ε T = prob $(1-\varepsilon)$

ξ_s = # flip till we get H

$P(\xi_s=1)=\varepsilon$, $P(\xi_s=2)=(1-\varepsilon)\varepsilon$

$E(\xi_s)=\frac{1}{\varepsilon}$

& $u^t(s)\le 1$ all s all ε

Claim with this ξ_s indep all s & t

$\Rightarrow \Pr(s^t=s^{t-1})\ge(1-\varepsilon)$

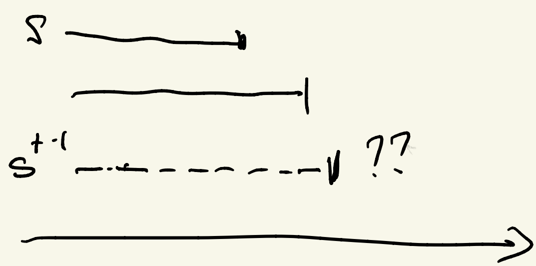
$$s^{t-1}=\arg\max\left[\sum_{\tau=1}^{t-1}u^\tau(s)+\xi_s\right]$$

Proof

consider event s^{t-1} was selected

& see outcomes all coins $s\neq s^{t-1}$

see outcomes s^{t-1} coins till we know it wins



With prob $1-\varepsilon$ next coin also T

$\Rightarrow s^{t-1}$ won by ≥ 1 amount

if $\boxed{u^t(s)=[0,1]} \Rightarrow$ this implies $s^t=s^{t-1}$

Real follow perturbed leader

ξ_s^t noise each iter & each s indep as above

used at time t $\tilde{s}^t$ " $s^{t-1} = \arg\max \sum_{\tau=1}^{t-1} u^\tau(s) + \xi_s^{t-1}$

$$E\left(u^\tau(s^{\tau-1})\right) \geq (1-\varepsilon) E\left(u^\tau(s^\tau)\right)$$

true prob on same ξ_s

& same dist on ξ_s

sum of τ

$$E\left(\sum_{\tau=}^{t} u^\tau(s^{\tau-1})\right) \geq (1-\varepsilon) E\left[\sum_{\tau=1}^{t} u^\tau(s^\tau)\right]$$

$$\geq (1-\varepsilon) \max_s E\left[\left(\sum_{\tau=1}^{t} u^\tau(s) + \xi_s\right) - \max_s \xi_s\right]$$

$$\geq (1-\varepsilon) E\left(\sum_{\tau=1}^{t} u^\tau(s)\right) - E\left(\max_s \xi_s\right]$$

Need:

$$E\left(\max_{s \in S} \xi_s\right) = O\left(\frac{\ln|S|}{\varepsilon} + 1\right)$$

Made with Goodnotes

choose $\varepsilon = \left(\ln |S| \cdot T\right)^{-1/2}$