Potential games & quality of solution

n players, player i sets $\mathcal{P}_i \subseteq 2^E$

player i strategy $P_i \in \mathcal{P}_i$

$f(e) = \# i$ s.t $e \in P_i$ all e

cost for player i $\quad cost_i(f) = \sum_{e \in P_i} c_e(f(e))$

Thm this is potential game

$$\phi(f) = \sum_e \sum_{k=1}^{f(e)} c_e(k)$$

Example: cost-sharing

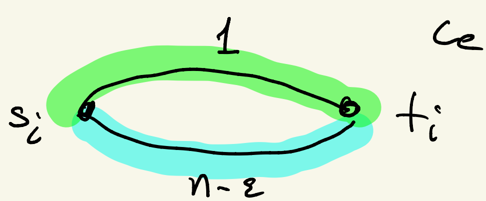
each element E has fixed cost c_e

$$c_e(k) = \frac{c_e}{k}$$

cost decreasing with congestion

Price of Anarchy

n users



Nash option 1: all upper edge

cost/user $= 1/n$

option 2: all down

cost/user $1 - \frac{\varepsilon}{n}$

Question: quality of best Nash:

Price of stability $= \min_{f \text{ Nash}} \frac{\text{cost}(f)}{OPT}$

<u>Claim</u>: $\arg\min_f \Phi(f)$ is a good Nash

$\text{cost}(f) \leq H_n \, OPT$ for cost sharing

<u>Proof</u>: (for cost sharing)

$$\text{cost}(f) = \sum_e f(e)\, c_e(f_e) = \sum_{e: f(e)>0} c_e$$

$$\Phi(f) = \sum_e \sum_{k=1}^{f(e)} c_e(k) = \sum_{e: f(e)>0} c_e + \frac{c_e}{2} + \dots \frac{c_e}{f(e)}$$

$$= \sum_{e: f(e)>0} c_e H_{f(e)}$$

$\text{cost}(f) \leq \Phi(f)$

$\Phi(f) \leq \text{cost}(f)\, H_n$

$H_k = 1 + \frac{1}{2} + \dots \frac{1}{k}$

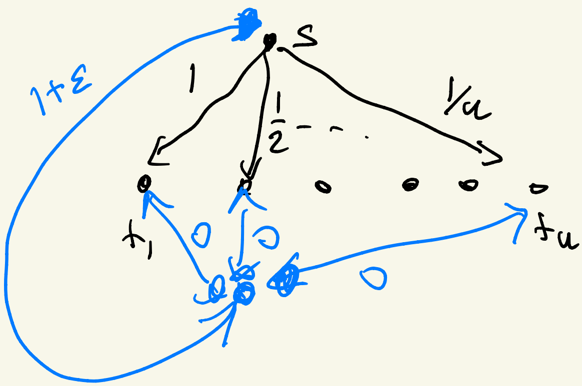
Made with Goodnotes

Proof: f^* min cost, f Nash minimizing ϕ

$$\mathrm{cost}(f) \le \phi(f) \le \phi(f^*) \le H_n \, \mathrm{cost}(f^*)$$

our choice of f

<u>Fact $H_n \sim \ln n$</u>



OPT = blue
total cost $1+\varepsilon$

black: unique Nash

Monday: routing $a_e x + b_e$ cost

f $$\mathrm{cost}(f) = \sum_e f_e(e)\left[a_e(f(e)) + b_e\right]$$

$$\phi(f) = \sum_e \sum_{k=1}^{f(e)} \left[a_e(k) + b_e\right]$$

$$\phi(f) \le \mathrm{cost}(f)$$

Example $a_e = 1$ $b_e = 0$ $f(e) = x$

cost for this edge x^2

ϕ — $(1$ — $\quad 1 + 2 + \dots + x = \frac{x(x+1)}{2} \geq \frac{x^2}{\cancel{2}}$

Claim
$$\phi(f) \geq \frac{1}{2} \operatorname{cost}(f)$$

~~Clai~~ Thm linear congestion routing
Price of stability ≤ 2

Proof: f^* opt f is ϕ minimizer

$$\operatorname{cost}(f) \leq 2\phi(f) \leq 2\phi(f^*) \leq 2\operatorname{cost}(f^*)$$