today routing
(atomic)

$n$ users $\quad s_i - t_i$ path $\to P_i$

$c_e(x)$ = fn with $x = \#$users

$$f(e) = \#\{i : e \in P_i\}$$

$$\text{cost}_P(f) = \sum_{e \in P} c_e(f(e))$$

Magic fact: $\exists$ a function $\emptyset$

$\emptyset$ potential fn:

$$\emptyset(P_1 \ldots P_n) - \emptyset(P_1 \ldots Q_i \ldots P_n)$$

$$= \text{cost}_i(P_1 \ldots P_n) - \text{cost}_i(P_1 \ldots Q_i \ldots P_n)$$

more generally. $s$ strategy vector

$$\emptyset(s) - \emptyset(s_i', s_{-i}) = c_i(s) - c_i(s_i', s_{-i})$$

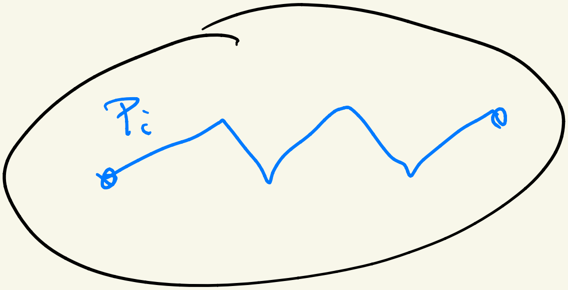
<u>Thm</u>: routing game above has potential fn. (any cost $c_e(x)$)

Proof: $$\emptyset(f) = \sum_e \sum_{k=1}^{f(e)} c_e(k)$$

recall $\text{cost}(f) = \sum_e f(e)\, c_e(f(e))$

$$\phi(f(P_1 \dots P_k)) - \phi(P_1 \dots \emptyset \dots P_k)$$

$P_i$ deleted




change term $c_e(f(e))$ missing from each $e \in P_i$

$$= \sum_{e \in P_i} c_e(f(e)) = \text{cost}_{P_i}(f)$$

same also applies

$$\phi(f(P_1 \dots Q_i \dots P_k)) - \phi(P_1 \dots \emptyset \dots P_k)$$

$$\text{cost}_{Q_i}(f(P_1 \dots Q_i \dots P_k)) \quad \blacksquare$$

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Corollaries

① repeated better response by players decreases potential

$\Longrightarrow$ does not cycle

there are finite options

$\Longrightarrow$ will end.

result Nash

② flow $f$ min $\Phi(f)$ is a Nash

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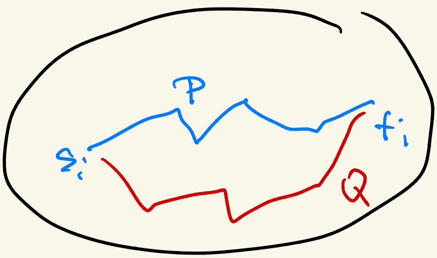
How about non-atomic flow?

$$\Phi(f) = \sum_e \int_0^{f(e)} c_e(x)\,dx = \sum_e \sigma_e(f(e))$$

Thm $f$ min $\Phi(f)$ is Nash.

$f_P > 0$




<u>Proof:</u> claim $c_P(f) \le c_Q(f)$

$$\sigma_e(y) = \int_0^y c_e(x)\,dx$$

tiny flow $P \to Q$ changes $\sum_e \sigma_e(f(e))$

$$\sum_{e\in Q} \sigma_e'(f(e)) - \sum_{e\in P} \sigma_e'(f(e))$$

$$= \sum_{e\in Q} c_e(f(e)) - \sum_{e\in P} c_e(f(e)) \ge 0$$

if $f$ min $\Phi$

Can we find $\min_{f} \Phi(f)$?

is $\sigma_e(y) = \int_0^y c_e(x)\,dx$ convex?

yes if $c_e(x)$ is monotone increasing

$\Longrightarrow$ (1) min ~~fix~~ can be found by convex optimization

(2) $f$ Nash iff only if minimizes $\Phi(f)$