today PoA on atomic routing

all users carry 1 unit of flow

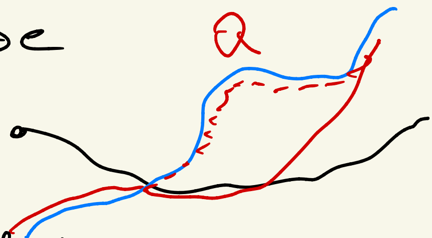
s_i & t_i source-sink pairs

Network, cost $c_e(x) \geq 0$ mon. increasing

user i selects path P_i from $s_i - t_i$

today: costs linear $c_e(x) = a_e x + b_e$

$f(e) = \# i$ s.t. $P_i \ni e$



cost of $P_i = \sum_{e \in P_i} c_e(f(e))$

total cost $\sum_i \text{cost}_{P_i}(f) = \sum_e f(e)\, c_e(f(e))$

$$OPT = \min_{P_i \,..\, \text{defining } f(e)} \sum_e f(e)\, c_e(f(e))$$

Nash: $P_1 \dots P_n$ if

$$\text{cost}_{P_i}(f) = \sum_{e \in P_i} c_e(f(e)) \le \sum_{e \in Q_i \cap P_i} c_e(f(e)) + \sum_{e \in Q_i \setminus P_i} c_e(f(e)+1)$$

all path Q_i from $s_i - t_i$

Try proof from last time:

$P_1 \dots P_n$ Nash $\longrightarrow f(e)$ flow one

$Q_1 \dots Q_n$ Optimal $\longrightarrow f^*(e)$ opt flow one

no user i wants to switch from $P_i \to Q_i$

We know

$$\sum_{e \in P_i} c_e(f(e)) \le \sum_{e \in Q_i} c_e(f(e)+1)$$

sum over users

$$\sum_i \sum_{e \in P_i} c_e(f(e)) \le \sum_i \sum_{e \in Q_i} c_e(f(e)+1)$$

$$\text{cost}(f) \le \sum_e c_e(f(e)+1) \cdot \underbrace{\sum_{i : e \in Q_i} 1}$$

$$\text{cost}(f) \le \sum_e c_e(f(e)+1) \cdot f^*(e) \qquad (*)$$

Magic fact for linear c fn.

$f(e) = x \qquad f^*(e) = y$

$$c(x+1)\, y \leq \frac{5}{3}\, y\, c(y) + \frac{1}{3}\, x\, c(x)$$

proof in a bit

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using this

continuing from *

$$\leq \frac{5}{3} \underbrace{\sum_e c_e(f^*(e))\, f^*(e)}_{\text{cost}(f^*)} + \frac{1}{3} \underbrace{\sum_e f(e)\, c_e(f(e))}_{\text{cost}(f)}$$

$$\frac{2}{3}\, \text{cost}(f) \leq \frac{5}{3}\, \text{cost}(f^*)$$

<u>Thm</u>: Price of anarchy for pure Nash on atomic flow with linear delays ≤ 2.5

Proof of magic fact: $c(x) = ax+b$

$$y(a(x+1)+b) \leq \frac{5}{3} y(ay+b) + \frac{1}{3} x(ax+b)$$

for all $a,b \geq 0$ & $x,y \geq 0$ integer

- obviously true on b part: $\frac{5}{3} > 1$

need: $a=1$, $b=0$ case

$$y(x+1) \leq \frac{5}{3} y^2 + \frac{1}{3} x^2$$

$$3yx + y \cdot 3 \leq 5y^2 + x^2$$

$$3y \leq (2y-x)^2 + y(x+y)$$

OK if $x+y \geq 3$ ✓

if $y = 0$ ✓

if $x+1 \leq y$ ✓

(x,y) can be $(1,1)$

General framework:

n players with cost $s_1 \dots s_n$ Nash

$s_1^* \dots s_n^*$ special

$$\sum_i \text{cost}_i(s) \underset{\text{Nash}}{\leq} \sum_i \text{cost}_i(s_i^*, s_{-i})$$

Nash

$$\underset{\text{magic fact}}{\leq} \lambda \, \text{Opt} + \mu \sum_i \text{cost}(s)$$

magic fact

if $\mu < 1 \implies \text{PoA} \leq \dfrac{\lambda}{1-\mu}$

[example $\lambda = 5/3$ $\mu = 1/3$)