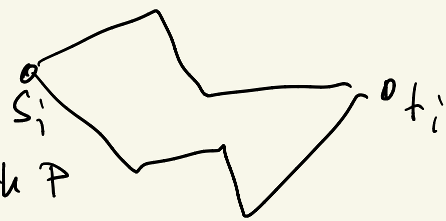
non-atomic flow

s_i, t_i rate r_i

flow from s_i to t_i carrying r_i flow

Network cost $c_e(x)$ edges

$c_e(x) \geq 0$ non increasing, cont.



$f_P \geq 0$ amount of flow on path P

$$\sum_{P: s_i \text{ to } t_i \text{ path}} f_P = r_i$$

total on edge e

$$f(e) = \sum_{P: e \in P} f_P$$

cost of P: $\displaystyle\sum_{e \in P} c_e(f(e)) = c_P(f)$

cost of f

$$\text{cost}(f) = \sum_P f_P \, c_P(f)$$

$$= \sum_P f_P \sum_{e \in P} c_e(f)$$

$$= \sum_{e} c_e(f) \cdot \sum_{P:\, e \in P} f_P = \sum_e c_e(f)\, f(e)$$

Equilibrium

all P with $f_P > 0$ s_i to t_i

$$c_P(f) = \min_{Q\ s_i \text{ to } t_i} c_Q(f)$$

Consider f ~~Nas~~ equilibrium flow

fixed edge cost $\bar{c}_e = c_e(f(e))$

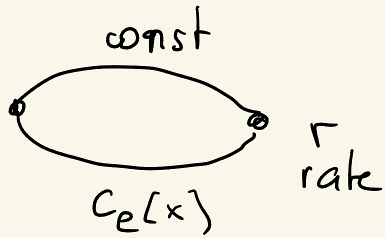
with $\bar{c}$ cost f is min cost

all other feasible flows f^*

cost f^* with $\bar{c}$ cost $\geq$ cost(f)

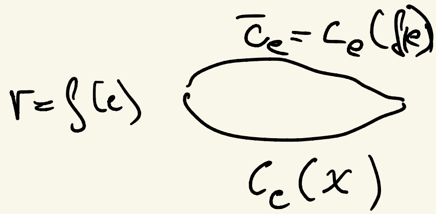
$\alpha(\mathcal{C}) = \max \dfrac{\text{Nash}}{\text{Opt}}$ on these ...

class of possible cost



Thm: all cost from $\mathcal{C}$ then PoA bounded by $\alpha(\mathcal{C})$

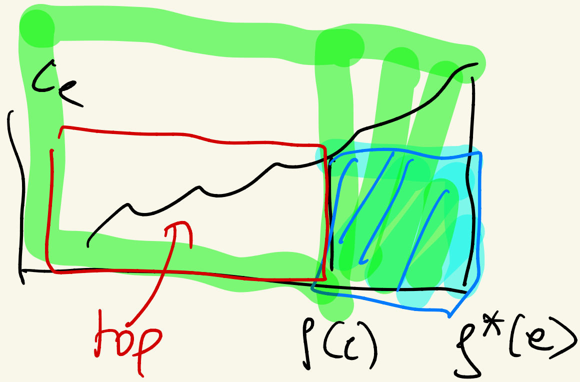
Proof: f^* – min ~~cos~~ social cost flow



Claim

$$\alpha \geq \frac{c_e(f(e))\, f(e)}{f^*(e)\, c_e(f^*(e)) + \bar{c}_e\,(f(e) - f^*(e))} \quad \leftarrow \text{equilibrium}$$

Note also true if $f^*(e) > f(e)$



adding for all e

$$\sum_e c_e(f(e))\, f(e) \leq$$

$$\alpha \left[\sum_e f^*(e)\, c_e(f^*(e)) \right] +$$

$$\alpha \sum_e \bar{c}_e \left(f(e) - f^*(e) \right)$$

$$\text{cost}(f) \le \alpha \left[OPT + \text{cost}(f) - \sum_e \bar{c}_e f^*(e) \right]$$

$$\le \alpha \left[OPT + \cancel{\text{cost}(f)} - \cancel{\text{cost}(f)} \right]$$

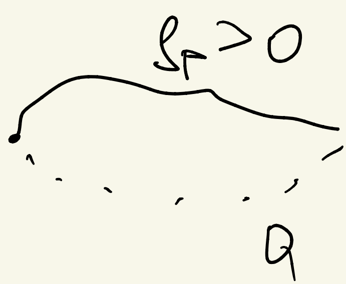
as with $\bar{c}$ cost
f is min cost

$$\Longrightarrow \text{cost}(f) \le \alpha\, OPT$$

· Intuition

recall eq. all flow on path
min cost $\sum_{e \in P} c_e(f(e))$

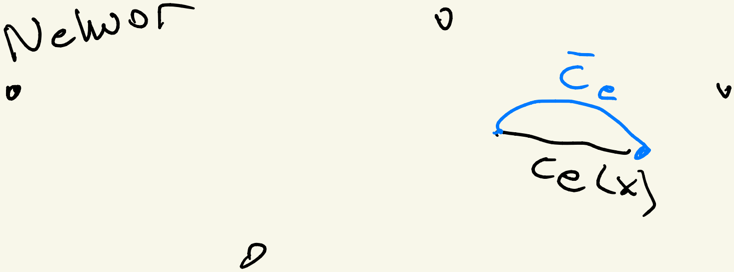
$\overline{\text{cost of flow}}$ $\quad \sum_e c_e(f(e)) \cdot f(e)$



if derivative of P
> derivative on Q
$\Longrightarrow$ global improves
moving bit of flow $P \to Q$

$$[c_e(x)\,x]' = \underbrace{c_e(x)}_{\text{selfish}} + x\,\underbrace{c_e'(x)}_{\text{social pain}}$$

Networ



all edges add new copy

Claim
- Nash unchanged
- Opt only improves

Opt on this = optimize edge by edge