# Strong Nash / collaboration

game: $n$ player $\{1, \dots, n\}$

strategies $S_i$ player $i$

Nash: all subsets $A \subseteq \{1, \dots, n\}$

& strategies $s_i'$

never the case

cost minimization game

$$c_i(s'_A, s_{-A}) \leq c_i(s) \quad \text{all } i \in A$$

& $\exists i \in A$ with $<$

[ $s_A$ = $s$ used by $i \in A$

$s_{-A}$ = $s$ used by $i \notin A$ ]

strong Nash $\boxed{= \text{coalition proof Nash}}$ ?

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trouble: may not exists!

e.g. prisoneer dilemma

costs

| | C | D |
|---|---|---|
| C | 1, 1 | 10, 0 |
| D | 0, 10 | 9, 9 |

Example: cost-sharing

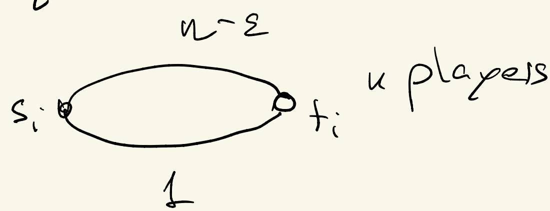
strategy $S_i \subseteq 2^E$ sets

$c_e \geq 0$ cost

if $x_e = \#$ players $e \in S_i$

$$c_i(S) = \sum_{e \in S_i} \frac{c_e}{x_e}$$

bad example




Claim: Costsharing

price of anarchy for strong Nash

$\leq H_n$

$$H_n = 1 + \frac{1}{2} + \dots \frac{1}{n}$$

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Proof: $S$ is Nash $S_1 \dots S_n$

Optimum: $S_1^* \dots S_n^*$

strong Nash $\Rightarrow \exists$ player, say $n$

$$c_n(S^*) \geq c_n(S)$$

$A_{n-1} = \{1, \dots, n-1\}$

$\exists$ player, say $n-1$

$c_{n-1}(s^*_{A_{n-1}}, s_n) \geq c_{n-1}(s)$

generally:

use numbering as above

$A_i = \{1, \dots, i\}$

$c_i(s^*_{A_i}) \geq c_i(s^*_{A_i}, s_{-A_i}) \geq c_i(s)$

$\uparrow$ cost for $i$ doing $s_1^* \dots s_i^* \ \emptyset \dots \emptyset$

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Summing

$$\text{cost}(s) = \sum_i c_i(s) \leq \sum_i c_i(s^*_{A_i}) \quad (*)$$

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$$c_i(s^*_{A_i}) = c_i(s^*_{A_i}) - c_i(s^*_{A_{i-1}})$$

Recall: cost-sharing potential game

$$c_i(s) - c_i(s_i', s_{-i}) = \phi(s) - \phi(s_i', s_{-i})$$

$s = s_i^* \quad s_i' = \emptyset$

Using this & $c_i(s^*_{A_{i-1}}) = 0$

$$c_i(s^*_{A_i}) = \phi(s^*_{A_i}) - \phi(s^*_{A_{i-1}})$$

continuing $\circledast$

$$= \sum_i \phi(s^*_{A_i}) - \phi(s^*_{A_{i-1}}) = \phi(s^*) - \phi(\phi)$$

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Recall $\phi$

- $$\phi(s) = \sum_{\substack{e \\ x_e > 0}} \sum_{k=1}^{x_e} \frac{c_e}{k} = \sum_{\substack{e \\ x_e > 0}} H_{x_e} c_e$$

$$\Longrightarrow \phi(\phi) = 0$$

- $$\text{cost}(s^*) = \sum_{x^*_e > 0} c_e \leq \phi(s^*) \leq H_n \text{cost}(s^*)$$

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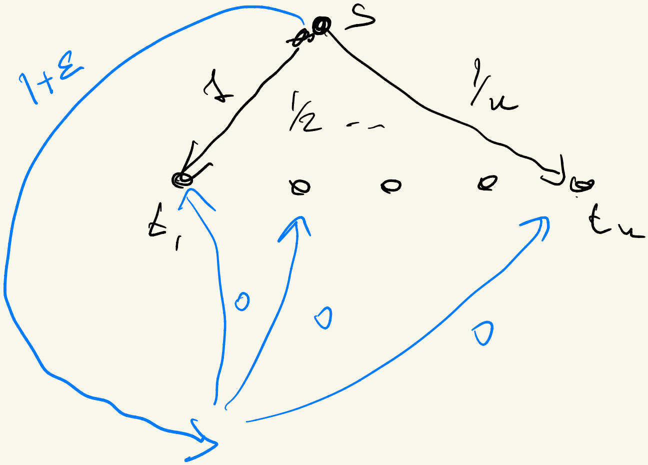
using this

we have $s$ Nash

$$\boxed{\text{cost}(s)} \leq \sum_i c_i(s) \leq \sum_i c_i(s^*_{A_i}) = \phi(s^*)$$

$$\leq \boxed{H_n \text{cost}(s^*)}$$

Recall other example: price of stability




$OPT = 1+\varepsilon$

Nash = strong Nash
black solution

$\Longrightarrow$ Price of anarchy bound is tight.