Price of anarchy analysis
via equilibrium analysis

So far: (1) (λ, μ) - smoothness
(2) equilibrium optimizes almost the right objective

Example: sharing resources
available resource $= 1$
player i value $v_i(x)$ for x amount
assumption:
$v_i(.)$ cont, diff, non increasing
& concave

utility
$v_i(x_i) - w_i$

$\Uparrow$

game: player i – submit w_i amount
get $x_i = \frac{w_i}{\sum w_j}$ all pay

Optimization for player i

$$w = \arg\max\ v_i\left(\frac{w}{w+\sum_{j\neq i} w_j}\right) - w_i$$

set der $= 0$

$$v_i'\left(\frac{w}{w+\sum_{j\neq i} w_j}\right)\cdot\frac{\sum_{j\neq i} w_j}{\left(w+\sum_{j\neq i} w_j\right)^2} = 1$$

$$\boxed{\frac{w}{w+\sum_{j\neq i} w_j} = 1-\frac{\sum_{j\neq i} w_j}{w+\sum_{j\neq i} w_j}}$$

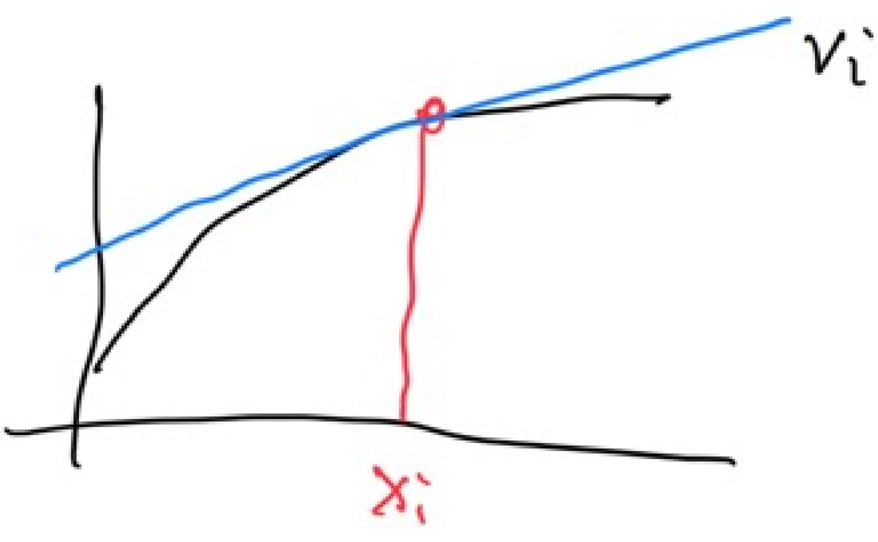
condition using better notation

$$v_i'(x_i)\,(1-x_i) = p$$

used $x_i = \frac{w_i}{\sum_j w_j}$, $p = \sum_j w_j$

Goal: worst case Nash vs Opt

Simplify v_i ~~to~~ making ratio wrt better



x_i Nash

change $\bar{v}_i(x) = v_i'(x_i)(x-x_i) + v_i(x_i)$

$\bar{v}_i(x) \geq v_i(x)$ & at x_i equal & equal derivative

Claim $\bar{v}_i(x)$ same Nash with same value

Opt with $\bar{v}_i$ $\geq$ Opt v_i

$\Longrightarrow$ worst case all $v_i(x) = a_i x + b_i$

$\bar{\bar{v}}_i(x) = a_i x$

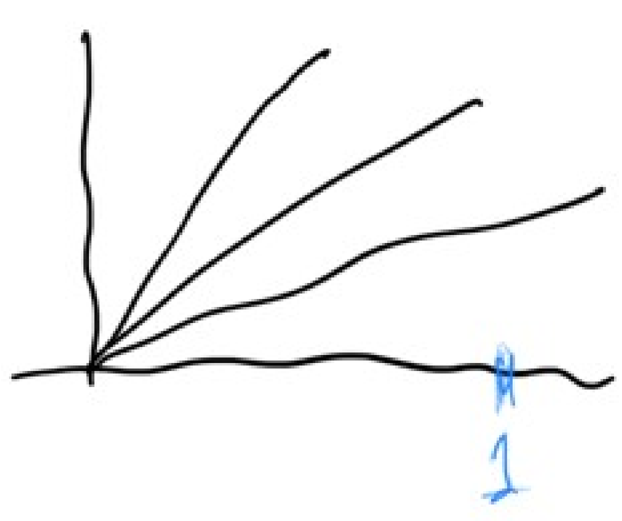
Note: $b_i \geq 0$ as $b_i = \bar{v}_i(0) \geq v_i(0) \geq 0$

$$\sum_i \bar{v}_i(x_i) - \sum_i \bar{\bar{v}}_i(x_i) = \sum_i b_i$$

- Opt: same solution $\sum b_i$ less value
- Nash: — " — $\sum b_i$ less value

$$\frac{\text{Nash}(\bar{v})}{\text{Opt}(\bar{v})} \geq \frac{\text{Nash}(\bar{\bar{v}})}{\text{Opt}(\bar{\bar{v}})} = \frac{\text{Nash}(\bar{v}) - \sum b_i}{\text{Opt}(\bar{v}) - \sum b_i}$$

$\Longrightarrow$ worst case: all $v_i(x) = a_i x$



$a_1 \geq a_2 \dots \geq a_n$

Opt $= x_1 = 1$

value $a_1 \cdot 1 = a_1$

Nash: $a_1(1-x_1) = p$

others $a_i(1-x_i) = p \Longrightarrow a_i > p$ if $x_i > 0$

Worst case $a_2(1-x_1) = p$

all others $a_i = p + \varepsilon$

all get x_i s.t. $a_i(1-x_i) = p$

Limit as $\varepsilon \to 0$ (#users $\to \infty$)

$a_1 x_1 + p\cdot(1-x_1)$ Nash value

$= a_1 x_1 + a_1(1-x_1)(1-x_1)$

worst case $x_1 = 1/2$

value $a_1 \frac{1}{2} + a_1 \frac{1}{4} = a_1 \frac{3}{4}$