No class on Wed.

project OK 4 people, larger goal.

Auction with XOS valuations

$$v_i(S) = \max_\ell \sum_{j \in S} v_{ij}^\ell$$

need to be able to do

1. given S find $v(S)$ & ℓ s.t.

$$v(S) = \sum_{j \in S} v_{ij}^\ell$$

2. given prices p_j for each item j

$$\arg\max_S \; v(S) - \sum_{j \in S} p_j$$

Claim: if list valuation $\ell \in \{1, \dots, k\}$

then easy to do in $O(km)$ m # items

Claim 1: in poly time given 1 & 2

we can get no-envy solution for bidding

in 2nd price auction

$$\sum_{t=1}^T u_i(s^t) \geq \max_S \sum_t \left(v_i(S) - \sum_{j \in S} p_j^t \right) - \text{Reg}$$

p_j^τ = max bid on item j at time τ

Claim 2

If all players satisfy no envy & there is no overbidding

all τ & S, & i $\quad v_i(S) \geq \sum_{j \in S} b_{ij}^\tau$

$\Longrightarrow$ $SW \geq \frac{1}{2} OPT - n\,Reg$

$n = \#$ players

Proof of Claim 2:

no overbidding

(*) $$\sum_j p_j^\tau = \sum_i \sum_{j \in A_i} p_j^\tau \leq \sum_i v_i(A_i^\tau) = SW^\tau$$

A_i^τ = set of items won by i

proving claim

$$\underbrace{\boxed{\sum_i \sum_\tau u_i(s^\tau)}}_{SW_{alg}} \geq \overbrace{\boxed{\sum_i \sum_\tau \Big(v_i(S_i^*)}}^{OPT} - \sum_{j \in S_i^*} p_j^\tau \Big) \quad -nReg$$

use for $S = S_i^*$ ~~set of~~ person i gets in opt

$$OPT \le \sum_i \sum_\tau v_i(S_i^*) \le \boxed{\sum_i \sum_\tau u_i(s^\tau)} + \boxed{\sum_\tau \sum_j p_j^\tau} + n\,Reg$$

$$OPT \le 2\,SW + n\,Reg$$

Claim 1 achiable without overbidding

today: at time t do the following

$$\text{find } S^t = \arg\max_S \; t\,v_i(S) - \sum_{\tau=1}^{t} \sum_{j \in S} p_j^\tau$$

real alg only does $t-1$ & add noise

$$\text{find } v_i(S^t) = \sum_{j \in S^t} v_{ij}^\ell$$

bid (v_{ij}^ℓ) for item j

Want to prove this has no-envy ($Reg = 0$)

* Claim

$$u_i(v_i^\ell, s_{-i}) \approx \sum_j (v_{ij}^\ell - p_j)^+ \ge \sum_{j \in S} (v_{ij}^\ell - p_j)$$

for any other bids s_{-i}, all sets S

Proof of main claim

base $\tau = 0$

$$\sum_{\tau=1}^{t-1} u_i(S^\tau) + u_i(S^t) \overset{i.h}{\geq} \sum_{\tau=1}^{t-1} \left(v_i(S^t) - \sum_{j \in S^t} p_j^\tau \right)$$

use $S = S^t$

$$+ \sum_{j \in S^t} \left(v_{ij}^{\ell} - p_j^t \right)$$

due to $*$

$$= \sum_{\tau=1}^{t} \left(v_i(S^t) - \sum_{j \in S^t} p_j^\tau \right)$$

by choice ℓ

$$\geq \max_S \sum_\tau \left(v_i(S) - \sum_{j \in S} p_j^\tau \right)$$

by choice of S^t