# Budgeted Auctions

$T$ rounds, 1 item per round, first-price auction

$n$ players, $i$: value $v_{it} \in [0,1]$, bid $b_{it}$, budget $B_i$

$i$ gets $S_i \subseteq [T]$, $S_i = \{t : b_{it} > \max_{j \neq i} b_{jt}\}$

$V_i = \sum_{t \in S_i} v_{it}$, $P_i = \sum_{t \in S_i} p_t$ $\to \max_j b_{jt}$

$$U_i = \begin{cases} V_i - P_i & \text{if } P_i \leq B_i \\ -\infty & \text{if } P_i > B_i \end{cases}$$

## Social Welfare

$$SW = \sum_i V_i$$

Example $n=2$, $T=1$ | $v_1 = 1$, $B_1 = \varepsilon$; $v_2 = 2\varepsilon$, $B_2 = 2\varepsilon$ | $SW^* = 1$, $SW = 2\varepsilon$

## Liquid Welfare

$$LW = \sum_i \min\{V_i, B_i\}$$

$$LW^* = \sum_i \min\left\{\sum_{t \in O_i} v_{it}, B_i\right\}$$

# No-Regret

$$U_i \geq \max_{\text{action}} \hat{U}_i(\text{action}) - \text{Reg}$$

action = shading multiplier $\lambda \in [0,1]$

$\rightsquigarrow$ bid $\lambda \cdot v_{it}$

$\hat{U}_i(\lambda)$ : bid $\lambda \cdot v_{it}$ until rem. budget $< 1$

$$U_i \geq \max_{\lambda \in [0,1]} \hat{U}_i(\lambda) - \text{Reg}$$

$$\hat{V}_i(\lambda) = \sum_{\substack{t:\text{rem.}\\ \text{budget} \geq 1}} v_{it} \mathbb{1}\{\lambda v_{it} > \max_{j \neq i} b_{jt}\}$$

$$\hat{P}_i(\lambda) = \sum_{\substack{t:\text{rem.}\\ \text{budet} \geq 1}} \lambda \cdot v_{it} \mathbb{1}\{\lambda v_{it} > \max_{j \neq i} b_{jt}\}$$

$$\hat{U}_i(\lambda) = \hat{V}_i(\lambda) - \hat{P}_i(\lambda) = (1-\lambda)\hat{V}_i(\lambda) = \frac{1-\lambda}{\lambda}\hat{P}_i(\lambda)$$

$$LW_i \overset{?}{\geq} LW_i^*$$

## $\hat{U}_i(\lambda)$

Case 1: rem. budget $< 1$ $\qquad \hat{P}_i(\lambda) \geq B_i - 1$

$$\hat{U}_i(\lambda) \geq \frac{1-\lambda}{\lambda} B_i - \frac{1-\lambda}{\lambda} \geq \frac{1-\lambda}{\lambda} LW_i^* - \frac{1-\lambda}{\lambda}$$

Case 2: rem. budget $\geq 1$

$$\hat{U}_i(\lambda) = \sum_t (1-\lambda)\, v_{it}\, \mathbb{1}\{\lambda v_{it} > \max_{j\neq i} b_{jt}\}$$

$$\geq \sum_{t\in O_i} (1-\lambda)\, v_{it}\, \mathbb{1}\{\lambda v_{it} > p_t\}$$

$$\geq (1-\lambda) \sum_{t\in O_i} \left(v_{it} - \frac{p_t}{\lambda}\right)$$

$$\geq (1-\lambda)\, LW_i^* - \frac{1-\lambda}{\lambda} \sum_{t\in O_i} p_t \quad \neq P_i$$

$$\hat{U}_i(\lambda) \geq (1-\lambda) LW_i^* - \frac{1-\lambda}{\lambda} \sum_{t\in O_i} p_t$$

$U_i$

Case 1 $\quad i\in N_1 \qquad V_i \geq B_i \quad \Rightarrow LW_i \geq LW_i^*$

Case 2 $\quad i\in N_2 \qquad V_i < B_i \quad \Rightarrow LW_i = V_i = U_i + P_i$

$$LW_i \geq P_i + (1-\lambda) LW_i^* - \frac{1-\lambda}{\lambda}\sum_{t\in O_i} p_t \quad \cancel{-\,Reg}$$

$$LW_i^* \leq \frac{1}{1-\lambda} LW_i - \frac{1}{1-\lambda} P_i + \frac{1}{\lambda}\sum_{t\in O_i} p_t$$

$$\xrightarrow[i\in N_2]{\text{sum}} \quad \sum_{i\in N_2} LW_i^* \leq \frac{1}{1-\lambda}\sum_{i\in N_2} LW_i - \frac{1}{1-\lambda}\sum_{i\in N_2} P_i + \frac{1}{\lambda}\sum_{i\in N_2}\sum_{t\in O_i} p_t$$

$$\frac{1}{\lambda}\sum_t p_t$$

$$\frac{1}{\lambda}\left(\sum_{i\in N_1} P_i + \sum_{i\in N_2} P_i\right)$$

$$P_i = \sum_{t\in S_i} p_t$$

$$\sum_{i\in N_2} LW_i^* \leq \frac{1}{\lambda}\sum_{i\in N_1} P_i + \frac{1}{1-\lambda}\sum_{i\in N_2} LW_i \cancel{- \left(\frac{1}{1-\lambda} - \frac{1}{\lambda}\right)\sum_{i\in N_2} P_i}$$

if $\lambda \geq \frac{1}{2}$

$$\sum_{i\in N_1} LW_i^* \leq \sum_{i\in N_1} LW_i$$

$$LW^* \leq \sum_{i\in N_1} LW_i + \frac{1}{\lambda}\sum_{i\in N_1} P_i + \frac{1}{1-\lambda}\sum_{i\in N_2} LW_i$$

$$\leq \left(1+\frac{1}{\lambda}\right)\sum_{i\in N_1} LW_i + \frac{1}{1-\lambda}\sum_{i\in N_2} LW_i$$

$$1+\frac{1}{\lambda} = \frac{1}{1-\lambda} \Rightarrow \lambda = \frac{\sqrt{5}-1}{2} \simeq 0.618$$

$$\Rightarrow LW^* \leq \frac{3+\sqrt{5}}{2} LW$$

$$\simeq 2.62$$