CS 6840 Algorithmic Game Theory Dec 9, 2024

Lecture 41: History of online search auctions and generalized second price

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1 History of online search auctions

Early Internet Advertising. Selling advertisements online started in 1994, using the way of selling newspaper ads. Advertisers paid flat fees to show their ads a fixed number of times (typically, 1,000 showings or impressions) on the platform with circulation.

Generalized First-Price Auctions. In 1997, Overture introduced selling ads per-click (only pay if someone clicks the ads). The webpage has ads slots from top to bottom. Overture sold the ads by first price auction and sorted by bids. The top slot was won by the highest bidder, who then paid *their own bid* upon each click; the second top slot was won by the second hightest bidder and paid the second highest bid price; etc. This is keyword based (people search keywords in the search box, and the ads will be shown accordingly).

This mechanism was unstable due to the fact that bids could be changed very frequently. For example, the following Figure 1 & 2 show that two competing companies bid for the ads. The companies alternate to be on top by raising the bids a little from \$6 to \$10. Then they bid down to \$6 again because now being on second top is better off. And they repeat this process. This leads to oscillations and instability—bidders continually adjust their bids in response to competitors to optimize their payoffs.

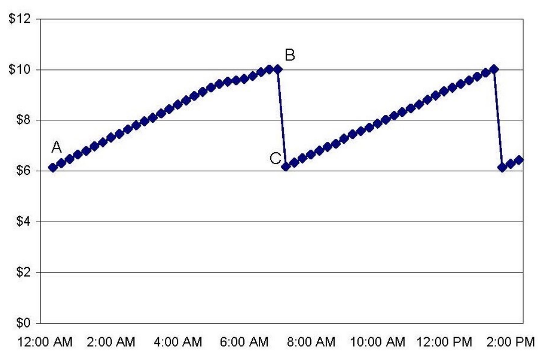

Figure 1: Example of daily fluctuations of bids in a first-price auction.

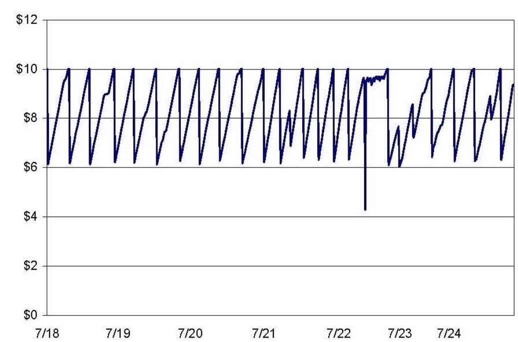

Figure 2: Longer-term fluctuations illustrating instability.

Generalized Second-Price Auction. In 2002, Google introduced the newly designed generalized second-price auction (GSP) mechanism. Yahoo!/Overture also switched to GSP. Each advertiser submits a bid. Advertisers are arranged on the page in descending order of their bids. The advertiser in the position i pays the bid of the advertiser in the position $i + 1$ as a price per click (or a reserve price if no lower bidder).

2 Generalized Second-Price Auction

Example: Let the values per click for the Advertisers be $v_1 = 10, v_2 = 4, v_3 = 2$. Suppose top slot has 200 clicks/hour, second slot has 100 clicks/hour.

If everyone bids truthfully, $b_1 = 10, b_2 = 4, b_3 = 2$:
- Top slot: Advertiser 1 pays 4 per click, payoff $= 200 \times (10 - 4) = 1200$.
- Next slot: Advertiser 2 pays 2 per click, payoff $= 100 \times (4 - 2) = 200$.

Fact: GSP is not truthful.

Now suppose top slot has 200 clicks/hour, second slot has 180 clicks/hour.

If Advertiser 1 lower their bid to 3, they might still secure a slot with 180 clicks but pay only 2 per click, yielding $180 \times (10 - 2) = 1440$, which is higher than 1200. Thus, *truth-telling is not a dominant strategy under GSP.*

3 Price of Anarchy for GSP

Assuming no overbidding (i.e., $b_i \leq v_i$ for all bids), we have:

Theorem 1 (PoA under GSP)**.** *If no advertiser bids above their valuation, then the Price of Anarchy (PoA) of GSP is at most 4 (More careful analysis refines this to about 3.16).*

Proof Sketch. Suppose bidding half of the value, $u_i(v_i/2, s_{-i}) \geq v_i/2 - max_j b_j$. Rest of proof as usual. ■

4 GSP and Generalized English Auction

In the generalized English auction, there is a clock showing the current price, which continuously increases over time. Initially, the price on the clock is zero, and all advertisers are in the auction. An advertiser can drop out at any time, and his bid is the price on the clock at the time when he drops out. The auction is over when the next-to-last advertiser drops out. The ad of the last remaining advertiser is placed in the best position on the screen, and this advertiser's payment per click is equal to the price at which the next-to-last advertiser dropped out. The ad of the next-to-last advertiser is placed second, and his payment per click is equal to the third-highest advertiser's bid, and so on. Simple English auctions sell one object.

Suppose there are k slots, the price of k^{th} slot is set when only k people left in the auction. What is the price of k^{th} slot? It turns out this auction is truthful.

Assume the sorted values per click to advertisers are $v_1 > ... > v_k > ... > v_n$.

Price of k^{th} slot: $p_k = v_{k+1}$ (when the bidder with value v_{k+1} drops out and only k bidders left).

Then the price keeps raising, what is the price of $k - 1^{th}$ slot? What is the moment the next people would want to drop out?

Assume the sorted fixed click rates are $\gamma_1 > ... > \gamma_k > \gamma_0$.

The next advertiser drops out if:
$\gamma_k(v - v_{k+1}) = \gamma_{k-1}(v - p)$ (i.e., The value he gets from the k^{th} slot = the value he gets from the $k - 1^{th}$

slot).
By v_k, the solution to this equation will set p_{k-1}.
The equation can be rearranged to be: $\gamma_{k-1}p_{k-1} = v_k(\gamma_{k-1} - \gamma_k) + \gamma_k v_{k+1}$.
$\gamma_{k-1}p_{k-1}$ is the price paid for slot $k-1$ and $\gamma_k v_{k+1}$ is the price paid for slot k.

This equation indicates that your price paid for slot $k-1$ is how much the other advertisers are suffering because of your presence. If the advertiser with v_{k-1} who won the slot $k-1$ were not present, the advertiser with v_{k+1} would get slot k instead of no slot and get the value $\gamma_k v_{k+1}$ from this slot; the advertiser with v_k would get slot $k-1$ instead of slot k and get extra value $v_k(\gamma_{k-1} - \gamma_k)$.

Therefore, in this equilibrium, each advertiser's resulting position and payoff are the same as in the dominant-strategy equilibrium of the game induced by Vickrey-Clarke-Groves (VCG)–each advertiser's payment is equal to the negative externality that he imposes on others, assuming that bids are equal to values.

Results:

1. Method is truthful (you should reveal your true value and drop out when you see there is no advantage of being in the game since the price will only increase).
2. Result is an equilibrium of GSP.
3. This equilibrium is envy-free: the advertisers do not want to be one slot higher or lower and do not envy their neighboring slot.

5 Envy-Free Equilibrium

Suppose b_i was the bid from advertiser i who won slot i.

If the advertiser who won slot i is envying the advertiser who won the slot below $(i+1)$, it implies that it is not an equilibrium and the advertiser i will bid just below b_{i+1}.

Advertiser i not envying $i-1$ requires that $\gamma_{i-1}(v_i - b_i) \leq \gamma_i(v_i - b_{i+1})$ where b_i is the price at slot $i-1$. This is true by rule setting prices.

Claim 1. *If the equilibrium is envy-free, then it is socially optimal. That is, $v_1 \geq \cdots \geq v_k \geq \ldots$, hence the solution maximizes social welfare.*

Proof. Let the price at slot i be p_i.
If envy-free, then advertiser i is not envying slot $i-1$: $\gamma_{i-1}(v_i - p_{i-1}) \leq \gamma_i(v_i - p_i)$.
If envy-free, then advertiser $i-1$ is not envying slot i: $\gamma_i(v_{i-1} - p_i) \leq \gamma_{i-1}(v_{i-1} - p_{i-1})$.
Summing the two inequalities: $\gamma_{i-1}v_i + \gamma_i v_{i-1} \leq \gamma_i v_i + \gamma_{i-1}v_{i-1}$
=> $v_i(\gamma_{i-1} - \gamma_i) \leq v_{i-1}(\gamma_{i-1} - \gamma_i)$
=> $v_i \leq v_{i-1}$. ∎

Claim 2. *If it is not envy-free, an advertiser i can continually increase their bid b_i until either becomes envy-free or the advertiser $i-1$ drops down.*

It is reasonable to expect that the solution will be envy-free.