CS 6840 Algorithmic Game Theory Dec 4, 2024

Lecture 29: Auctions with Budgets

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1 Introduction to Budgeted First-Price Auctions

We consider a setting with T rounds, where one item is auctioned in each round using a first-price auction. The participants in the auction are n players, each characterized by:

- a value $v_{it} \in [0, 1]$ representing player i's value for the item in round t.
- a bid $b_{it} \in [0, 1]$ made by player i in round t.
- a budget B_i which is the maximum amount player i can spend.

The auction mechanism involves each player bidding an amount b_{it} in every round t. The player with the highest bid wins the item for that round, but they must pay their bid amount. Hence, the bidding strategy and budget constraints heavily influence the outcome.

Let $S_i \subseteq [T]$ denote the set of rounds where player i wins the item. Formally,

$$S_i = \{t : b_{it} > \max_{j \neq i} b_{jt}\}.$$

We define the following quantities:

- **Value of player** i**:** $V_i = \sum_{t \in S_i} v_{it}$, representing the total value derived from the items won by player i.
- **Payment of player** i**:** $P_i = \sum_{t \in S_i} b_{it}$, representing the total payment made by player i across all winning rounds.

The utility U_i of player i is defined as:

$$U_i = \begin{cases} V_i - P_i, & \text{if } P_i \leq B_i, \\ -\infty, & \text{if } P_i > B_i. \end{cases}$$

This utility function is designed to discourage players from overspending since exceeding the budget results in a penalty (of negative infinity).

1.1 Social Welfare

The social welfare (SW) is defined as the total value derived by all players:

$$\text{SW} = \sum_i V_i.$$

Social welfare measures of the overall efficiency of the allocation mechanism, particularly, how well the items are allocated to players who value them the most. However, social welfare doesn't take the budgeted nature of the auctions into account. This is captured by the following example.

1.1.1 Example

- Number of players: $n = 2$
- Number of rounds: $T = 1$
- Values: $v_1 = 1$, $v_2 = 2\varepsilon$
- Budgets: $B_1 = \varepsilon$, $B_2 = 2\varepsilon$

In this case, player 1 has a higher value, but due to budget constraints, player 2 may win the auction depending on their bid. The optimal social welfare is $\text{SW}^* = 1$, while the realized welfare is $\text{SW} = 2\varepsilon$ under certain bidding strategies. This motivates a relaxed notion of welfare.

1.2 Liquid Welfare

Liquid welfare is an alternative measure defined as:

$$\text{LW} = \sum_i \min\{V_i, B_i\}.$$

This measure captures the welfare that respects budget constraints more directly than standard social welfare. It considers the minimum of the value derived and the budget available for each player.

The optimal liquid welfare is:

$$\text{LW}^* = \sum_i \min\left\{\sum_{t\in O_i} v_{it}, B_i\right\},$$

where O_i denotes the optimal set of items allocated to player i. This definition accounts for the fact that even if a player values certain items highly, their budget may limit their ability to bid for them effectively.

2 No-Regret Strategies

Players aim to maximize their utility while minimizing regret. We'll define the player's action be a shading multiplier $\lambda \in [0, 1]$, where the bid is defined as λv_{it}.

Shading allows players to strategically underbid based on their budget and expectations of other players' bids. The shading multiplier will try to balance the probability of winning with the cost of bidding.

We define the total value of player i under the shading multiplier as

$$\hat{V}_i(\lambda) = \sum_{t:\text{remaining budget}\geq 1} \left[v_{it}\mathbf{1}\{\lambda v_{it} > \max_{j\neq i} b_{jt}\}\right]$$

representing the total value derived from the items they won. Similarly we define the total payment as follows:

$$\hat{P}_i(\lambda) = \sum_{t:\text{remaining budget}\geq 1} \left[\lambda v_{it}\mathbf{1}\{\lambda v_{it} > \max_{j\neq i} b_{jt}\}\right]$$

Therefore the utility under a shading multiplier is given by:

$$\hat{U}_i(\lambda) = \hat{V}_i(\lambda) - \hat{P}_i(\lambda),$$

The no-regret condition requires:

$$U_i \geq \max_{\lambda \in [0,1]} \hat{U}_i(\lambda) - \text{Regret}.$$

This means that the player's utility under their chosen strategy is at least as good as the utility under the best alternative strategy, minus some regret term that diminishes over time.

3 Key Results and Bounds

We proceed to analyze how well we can approximate the optimal liquid welfare using a no-regret learner.

Theorem 1. *Consider the budgeted first-price auction setting described above, with T rounds and n players. Each player i has per-round values $v_{it} \in [0,1]$ and a total budget B_i. Suppose the players have outcomes with no regret (not even small regret). Then, there exists a choice of bidding parameters (specifically a shading multiplier λ) ensuring that the expected liquid welfare achieved satisfies:*

$$LW^* \leq \frac{3+\sqrt{5}}{2} \cdot LW,$$

where LW^* is the optimal liquid welfare and LW is the liquid welfare achieved by the no-regret dynamics.

To prove the result above, we start by lower bounding $\hat{U}_i(\lambda)$ for a player i. At the end of the auction we have one of the two cases:

Case 1: Remaining budget < 1

If the remaining budget is less than 1, this means that $\hat{P}_i(\lambda) \geq B_i - 1$

$$\hat{U}_i(\lambda) \geq \frac{1-\lambda}{\lambda} B_i - \frac{1-\lambda}{\lambda} \geq \frac{1-\lambda}{\lambda} \text{LW}^* - \frac{1-\lambda}{\lambda}.$$

Case 2: Remaining budget ≥ 1

The utility under a shading multiplier λ can be written as:

$$\hat{U}_i(\lambda) = \sum_t (1-\lambda) v_{it} \mathbf{1}\{\lambda v_{it} > \max_{j \neq i} b_{jt}\}.$$

Let O_i denote the set of rounds won by player i in the optimal liquid welfare and $p_t = \max_j b_{jt}$.

$$\hat{U}_i(\lambda) \geq \sum_{t \in O_i} (1-\lambda) v_{it} \mathbf{1}\{\lambda v_{it} > p_t\}.$$

Rearranging terms and using the fact that $x \cdot \mathbf{1}\{x > \epsilon\} \geq x - \epsilon$ to lower bound the RHS, we have

$$\hat{U}_i(\lambda) \geq (1-\lambda) \sum_{t \in O_i} \left(v_{it} - \frac{p_t}{\lambda} \right).$$

Thus, we obtain:

$$\hat{U}_i(\lambda) \geq (1-\lambda)\mathrm{LW}_i^* - \frac{1-\lambda}{\lambda} \sum_{t \in O_i} p_t.$$

Now to relate the liquid welfare to the optimal liquid welfare, we consider two cases:

Case 1: $i \in N_1$, the set of players for which $V_j \geq B_j$

For players in set N_1, the total value derived by player i exceeds their budget. Therefore, their liquid welfare is capped by their budget:

$$\mathrm{LW}_i \geq \mathrm{LW}_i^*.$$

Case 2: $i \in N_2$, the set of players for which $V_j < B_j$

For players in set N_2, the value derived by player i is less than their budget. In this case, we analyze their utility:

$$\mathrm{LW}_i = V_i = U_i + P_i.$$

From here, we use the no-regret guarantee and plug in the bound on utility from the case where the budget is greater than 1:

$$\mathrm{LW}_i \geq P_i + (1-\lambda)\mathrm{LW}_i^* - \frac{1-\lambda}{\lambda} \sum_{t \in O_i} p_t.$$

Summing the inequalities over all players in N_2, we obtain the following:

$$\sum_{i \in N_2} \mathrm{LW}_i^* \leq \frac{1}{1-\lambda} \sum_{i \in N_2} \mathrm{LW}_i - \frac{1}{1-\lambda} \sum_{i \in N_2} P_i + \frac{1}{\lambda} \sum_{i \in N_2} \sum_{t \in O_i} p_t.$$

Now we use the fact that $\sum_{i \in N_2} \sum_{t \in O_i} p_t \leq \sum_t p_t = \sum_{i \in N_1} P_i + \sum_{i \in N_2} P_i$, to reduce to the following

$$\sum_{i \in N_2} \mathrm{LW}_i^* \leq \frac{1}{1-\lambda} \sum_{i \in N_2} \mathrm{LW}_i - \frac{1}{1-\lambda} \sum_{i \in N_2} P_i + \frac{1}{\lambda} \sum_{i \in N_1} P_i + \frac{1}{\lambda} \sum_{i \in N_2} P_i.$$

Now assuming $\lambda \geq \frac{1}{2}$, we can remove the $\sum_{i \in N_2} P_i$ terms because their sum becomes negative

$$\sum_{i \in N_2} \mathrm{LW}_i^* \leq \frac{1}{1-\lambda} \sum_{i \in N_2} \mathrm{LW}_i + \frac{1}{\lambda} \sum_{i \in N_1} P_i.$$

Summing together the equation from Case 1 and using the fact that $P_i \leq \mathrm{LW}_i$, we get

$$\mathrm{LW}^* \leq \left(1 + \frac{1}{\lambda}\right) \sum_{i \in N_1} \mathrm{LW}_i + \frac{1}{1-\lambda} \sum_{i \in N_2} \mathrm{LW}_i.$$

By solving for λ that minimizes the bound, we find:

$$\lambda = \frac{\sqrt{5}-1}{2} \approx 0.618.$$

Thus, the final competitive ratio is:

$$\mathrm{LW}^* \leq \frac{3+\sqrt{5}}{2} \cdot \mathrm{LW} \approx 2.62 \cdot \mathrm{LW}.$$

This shows a competitive ratio of 2.62 for the liquid welfare relative to its optimal value under budget constraints.