CS 6840 Algorithmic Game Theory Friday, August 30th, 2024

Lecture 3: Hotelling games

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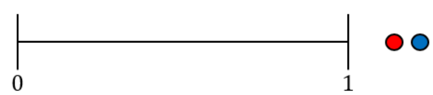
1 Hotelling games

In the previous classes, we talked about that the Braess paradox is not that bad. The easiest game to start proving how much we're losing is the Hotelling games.

Hotelling games have many variations, we're going to check the most famous version.

1.1 Ice cream provider problem

The problem is an ice cream provider that is deciding where to put its stand in a beach front. In our analysis, we need to include the assumption that beachgoers are densely populated throughout the entire area. We can represent this by considering them as points on a contiguous [0, 1] line in the beach.



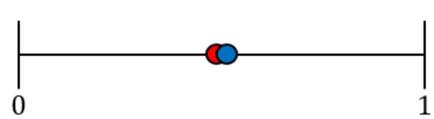
Let's assume that the customers will go to the closest stand. So no matter where the stand is located, people will come. So, if there's only one ice cream provider, it doesn't matter where the stand is positioned.

Consider that there are two ice cream providers that are trying to decide where to put their stands, let's call them the red guy and the blue guy.

The objective function of the providers in this problem is to maximize the # of customers served. Where they should locate their stands? What is the Nash equilibrium?

- If one guy is already located in the line, the other may think that moving further away can increase the number of customers. But, by moving further away (in any particular direction), the amount is increased in the section that we are moving to and reduced in the section that we are moving away from.
- The solution is to put both stands in the middle, right next to each other. The unique Nash equilibrium is that both stands are in the center.

Comment: Starbucks use a similar algorithm to locate their shops by locating their stores as close to other coffee shops as possible.

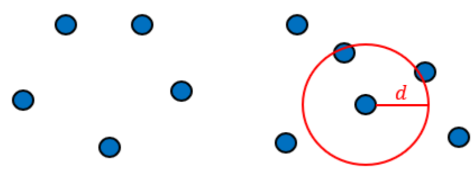


What about 3 players? Is there a unique Nash equilibrium? Is a Nash equilibrium to have all three players in the center?

If all of three players are in the center, one of them has incentives to move to one side, because it can increase the amount of customers served by proximity. When the players show their position (strategy), no matter where they are, somebody wants to move. So, there's not a Nash equilibrium.

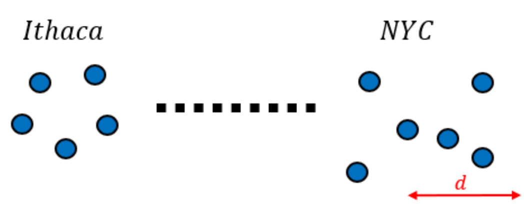
How can there be no Nash equilibrium? Didn't John Nash win a Nobel Prize for proving that every game has at least one Nash equilibrium? His theorem does have some conditions, though: the game needs a finite number of players, and each player must have a finite number of strategies. In our case, the second condition is not met—our three players have a continuously many possible strategies.

Let's create a finite version of this problem. Suppose that we have N fixed locations available, where each location v has w_v customers. Every pair of locations x and y are at a distance $d(x, y) \geq 0$. Every customer chooses the stand that is located as close as possible to them, with a maximum disposition to travel d. So if the distance is too big, then the customers don't go to any store.



Where is the best location to put a stand? This answer depends of d and the amount of customers of each node.

Suppose we have two players, and a set of locations in two far away cities such that nobody wants to travel from one city to the other. Let's call these cities as Ithaca and New York City (NYC). What is best strategy for the two providers?



Given the high amount of people on NYC , the best strategy is that both players are located in that city.

Let's define the social welfare as the total number of the customers served. Then, the social optimum of this problem as serving coffee as much people as we can.

$$\text{Social optimum} = \max \#\text{customers served}$$

Given an appropriate choice of d such that one player can collect all of the customers in either of two cities, and the fact that the distance between Ithaca and NYC is too big, we claim that the social optimum is one must be located in Ithaca and one must be in NYC.

$$\text{Social optimum} = \text{Max. } \#\text{customers served by locating one provider in NYC, one in Ithaca.}$$

What is the loss of efficiency if both players are in NYC? First, we define the utility for each player as:

$$\text{Utility for player } i := u_i = \#\text{Customers within a dist.} \leq d \text{ for whom this is the closest option.}$$

Assume that if two or more locations are at a equal distance, customers divide equally between them.

We care in this problem about the customers being served. We are going to call this concept as **social welfare**. The social welfare of this problem is:

$$\text{Social welfare} \rightarrow SW = \sum_i u_i = \#\text{ Customers served.}$$

The price of not controlling the positions of the providers, by not telling them what to do, is the **price of anarchy**:

$$\text{Price of anarchy:} = \frac{SW \text{ optimum solution}}{SW \text{ Nash equilibrium}}$$

Note that the SW in the optimal has a single value, while the SW in the Nash equilibrium may not be unique. Let's find the maximum price of anarchy over all possible Nash equilibriums, and see that this number is not arbitrarily bad.

$$\max \frac{SW \text{ optimum solution}}{SW \text{ Nash equilibrium}}$$

Observations:

- *If this number is (arbitrarily) bad, then a set of rules should be set. For example, limiting the stands located in NYC by giving permits. This intervention can cause other issues.*
- *This definition of social welfare is very insensitive to fairness, so frequently people in Ithaca may not be served.*

Suppose that we have K players, N locations, w_v customers per location v, and the max. distance is d. Suppose that $(s_1, \ldots, s_N) = S$ is a pure Nash equilibrium with the utilities $u_i(S)$ for the player i.

Proposed exercise: Proof that the Nash equilibrium in this case is not that bad.

The op. social optimum is $S^* = (s_1^*, \ldots, s_N^*)$, let's consider the utilities $u_i(s_i^*, s_{-i})$. We know that, because S is the Nash equilibrium:

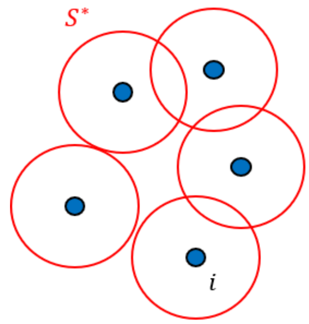
$$u_i(S) \geq u_i(s_i^*, s_{-i}) \tag{1}$$

Why don't the players want to do the optimal thing? To answer this, we need to study $u_i(s_i^*, s_{-i})$ in this solution:

$$u_i(s_i^*, s_{-i}): \quad \text{all customers within a dist} \leq d \text{ to } s_i^* \text{ are being served}(*).$$

(*): These customers may not be necessarily served by s_i^*, they may have others in s_{-i} that are closer.

In the following figure, the optimal solution S^* is represented with red circles. Since all locations are within these circles, everyone is served—either by player i or another player.



Let's consider the following notation:

- X_i^* is the set of customers served by i in the optimum.
- X_i is the set of customers served by i at Nash equilibrium.
- $\left|X_i^* \setminus \bigcup_j X_j\right|$ is the amount of people that is served by i in the optimum solution, and not served by anyone in the Nash equilibrium.

We claim that:

$$u_i(s_i^*, s_{-i}) \geq \left|X_i^* \setminus \bigcup_j X_j\right| \text{ (at least)}$$

Once we add s_i^*, the customers in X_i^* will have a location within distance d, ensuring they will be served.

$$\implies \sum_i u_i(s_i^*, s_{-i}) \geq \sum_i \left|X_i^* \setminus \bigcup_j X_j\right|$$

Since the sets described in the right side are disjoint, this sum represents the number of people served in the optimal scenario but not in the Nash equilibrium. Therefore:

$$\implies \sum_i u_i(s_i^*, s_{-i}) \geq SW(S^*) - SW(S)$$

As we noted in (1), the left side is always worse than the Nash. Then we can say that:

$$SW(S) = \sum_i u_i(S) \geq \sum u_i(s_i^*, s_{-i}) \geq SW(S^*) - SW(S)$$
$$\implies SW(S) \geq \frac{1}{2} SW(S^*)$$

Theorem: In this problem, the price of anarchy for a pure Nash equilibrium is at most a factor of 2.

Proposed exercise: Note that this result is limited to pure Nash equilibriums. Extend this result to mixed Nash equilibrium. Check the price of anarchy when randomness is applied and if no one wants to move.

Final remark: This is the standard recipe to compare the optimum solution and the Nash equilibrium.