**CS 6840 Algorithmic Game Theory** August 28, 2024

# Lecture 2: Game Theory Basics

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In this lecture, we will define a few basic concepts in game theory and illustrate with some examples.

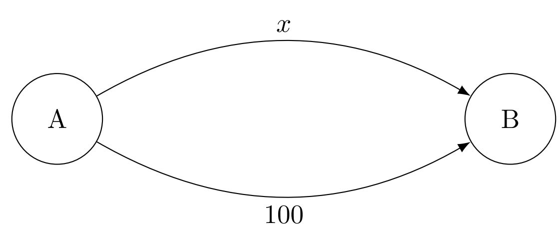
**Definition 1.** A game consists of:

- $N = \{1, \ldots, n\}$: set of players.
- $S_i$: set of pure strategies of player $i$.
- $s = (s_1, \ldots, s_n)$ collects each player's strategy. We call it a 'profile' of strategies.
- $S = \Pi_{i=1}^{N} S_i$: set of profiles of strategies.
- $u_i : S \to \mathbb{R}$: utility of player $i$. Alternatively, $c_i(s) = -u_i(s)$ is the cost of player $i$.

*Remark.* Importantly, $u_i$ not only depends on $s_i$ but the entire profile of strategies $s$. This is the essence of game theory – one's payoff depends on other players' actions.

*Remark.* We will use the convention that the objective of the players is to maximize their utilities, or equivalently, minimize their costs.

*Example* 1 (Routing Game). Consider the traffic routing game we introduced in lecture 1.[1]




100 cars want to get from A to B. The up route is congestion-sensitive and takes $x$ minutes to get through, where $x$ is the number of cars on the road. The down route is a highway that takes 100 minutes to get through regardless the number of cars. Agent's cost is the minutes of delay. Formulate this game with formal definitions.

- $N = \{1, \ldots, 100\}$.
- $S_i = \{U, D\}\ \forall i \in N$.
- $\forall i \in N, c_i(U) =$ number of cars choosing $U$, and $c_i(D) = 100$.

**Definition 2** (Pure Nash equilibrium)**.** A profile of strategy $s = (s_1, \ldots, s_n)$ is a (pure) Nash equilibrium if for all players $i$ and strategies $s_i' \in S_i$,

$$u_i(s) \geq u_i(s_i', s_{-i})$$

where $s_{-i}$ denotes the strategy of all players other than $i$. In other words, holding the other players' strategy fixed, no player can **strictly** benefit from deviating to another pure strategy.

[1] The diagram is created with reference to Cristian Palma's scribe notes.

*Remark.* In Example 1, one pure Nash equilibrium is $s_i = U, \forall i$ (all players go up). The other pure Nash equilibrium is 99 players go up and one goes down.

*Example* 2 (Rock-Paper-Scissor). We can represent the well-known game of Rock-Paper-Scissor using the following payoff matrix (the entries represent the row player's payoff since the game is zero-sum):

| | $R$ | $P$ | $S$ |
|---|---|---|---|
| $R$ | $0$ | $-1$ | $+1$ |
| $P$ | $+1$ | $0$ | $-1$ |
| $S$ | $-1$ | $+1$ | $0$ |

This game has no pure Nash equilibrium, because the losing player always have a strictly profitable deviation. This game does, however, admit a mixed Nash equilibrium where each player plays the three strategies with $\frac{1}{3}$ probability.

**Definition 3** (Mixed strategy)**.** A mixed strategy of player $i$ is a probability distribution over $i$'s pure strategies. Let $\Delta(S_i)$ be the probability simplex over $S_i$. A mixed strategy $\sigma_i \in \Delta(S_i)$ satisfies $\sum_{s_i} \sigma_i(s_i) = 1$, where $\sigma_i(s_i)$ denotes the probability mass assigned to the pure strategy $s_i$.

**Definition 4.** Let $\sigma = (\sigma_1, \dots, \sigma_n) \in \Pi_{i=1}^N \Delta(S_i)$ denote a profile of mixed strategies.

**Definition 5** (Mixed Nash equilibrium)**.** A profile of mixed strategy $\sigma = (\sigma_1, \dots, \sigma_n)$ is a mixed Nash equilibrium if for all players $i$ and pure strategies $s_i' \in S_i$,

$$\mathbb{E}_{s\sim\sigma}[u_i(s)] \geq \mathbb{E}_{s_{-i}\sim\sigma_{-i}}[u_i(s_i', s_{-i})] \tag{1}$$

*Remark.* A 'stronger' version of the definition could allow player $i$ deviate not only to any pure strategies but to any mixed strategies. These two definitions turn out to be equivalent.

*Claim* 3. If (1) is true for all $s_i' \in S_i$, it is also true for any probability distribution over $S_i$.

**Proof.** Let $\sigma_i' \in \Delta(S_i)$ be a probability distribution over $S_i$. Player $i$'s expected utility from playing the mixed strategy $\sigma_i'$ when the other players' strategy is fixed at $\sigma_{-i}$ is:

$$\begin{aligned} \mathbb{E}_{s_i'\sim\sigma_i'}[\mathbb{E}_{s_{-i}\sim\sigma_{-i}}[u_i(s_i', s_{-i})]] &= \sum_{s_i'\in S_i} \mathbb{E}_{s_{-i}\sim\sigma_{-i}}[u_i(s_i', s_{-i})] \cdot \sigma_i'(s_i') \\ &\leq \sum_{s_i'\in S_i} \mathbb{E}_{s\sim\sigma}[u_i(s)] \cdot \sigma_i'(s_i') \\ &= \mathbb{E}_{s\sim\sigma}[u_i(s)] \end{aligned}$$

In words, due to the linearity of expectations, utility of playing $\sigma_i'$ can be written as a weighted sum of utilities of playing $s_i'$. If (1) is true for all $s_i' \in S_i$ it must be true for any linear combination of $s_i'$. ■

*Example* 4 (Traffic Game). Two cars arrive at an intersection without traffic lights.

| | Wait | Pass |
|---|---|---|
| Wait | $0, 0$ | $0, 1$ |
| Pass | $1, 0$ | $-9, -9$ |

It is easy to see that this game has two pure Nash equilibria: (Wait, Pass) and (Pass, Wait). This game also has a mixed Nash equilibrium which can be calculated as follows.

Suppose that the row player plays 'Pass' with probability $x$ and that the column player plays 'Pass' with probability $y$. Consider the row player's payoff of deviating to any of her pure strategies.

The payoff of playing 'Wait' when the column player mixes is 0. The payoff of playing 'Pass' when the column player mixes is $1(1-y) - 9y = 1 - 10y$. According to Definition 5, a mixed Nash equilibrium requires that players do not have incentive to deviate to any pure strategies. This is only possible when the payoff from 'Wait' and 'Pass' are equal, because otherwise any mixing would be inferior to playing the pure strategy with the higher payoff.

Let $0 = 1 - 10y$, we can solve that $y = \frac{1}{10}$. The same reasoning for column player's deviations gives that $x = \frac{1}{10}$.

*Remark* (Randomize out of indifference)*.* In a mixed Nash equilibrium, the players must be indifferent between playing any pure strategies on which they put positive probability.

*Remark.* Note that we solved for the **row** player's mixing probability when considering the **column** player's deviations. This uncovers an important feature of mixed Nash equilibria – players choose the specific mixing probability only to make the other player indifferent.