CS 6840 Algorithmic Game Theory February 10, 2020

Lecture 8: Price of Anarchy in Nonatomic Flows

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We will continue to consider how smoothness methods can be used to derive Price of Anarchy bounds. In this lecture, we will specifically consider the setting of nonatomic flows.

1 Nonatomic Flows

Recall that in the nonatomic flow setting, each edge e will be associated with a delay function d_e and each player is trying to minimize their cost function $\sum_{e \in P} d_e$ for the path P they choose. We define the social cost to be:

$$SC(f) = \sum_P f_P \sum_{e \in P} d_e(f(e)) = \sum_e f(e) d_e(f(e)) \tag{1}$$

where f_P denotes the flow through path P for f.

We will denote a given Nash equilibrium by f and the optimal solution (`OPT`) f^*. By the definition of a Nash equilibrium, we know that for each player, the following holds:

$$\sum_{e \in P} d_e(f(e)) \leq \sum_{e \in P^*} d_e(f(e)) \tag{2}$$

where P indicates the path they choose in f and P^* indicates the path in f^*.
Summing the inequality in Equation 2 across all players and rewriting as a sum over edges (as we have seen before) yields:

$$\sum_e f(e) d_e(f(e)) \leq \sum_e f^*(e) d_e(f(e)) \tag{3}$$

Observe that the LHS in Equation 3 is precisely $SC(f)$.

As we will expand upon next lecture, the following type of generalized magic/smoothness inequality will imply a Price of Anarchy bound:

$$\forall e, \;\; f^*(e) d_e(f(e)) \leq \lambda f^*(e) d_e(f^*(e)) + \mu f(e) d_e(f(e)) \tag{4}$$

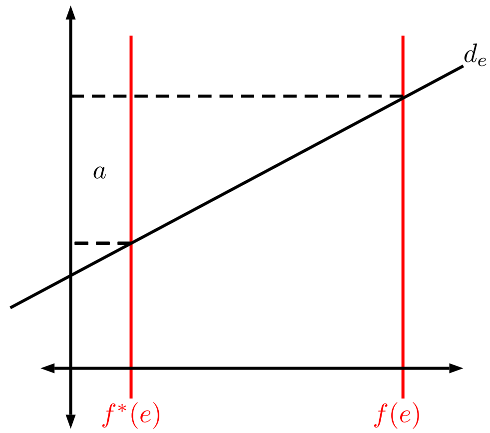
for constants $\lambda, \mu > 0$, $\mu < 1$. We begin by demonstrating this can be done in the case of linear delays.

1.1 Linear Delays

Theorem 1. *If $d_e(\cdot)$ is a linear function, then for any edge e,*

$$f^*(e) d_e(f(e)) \leq f^*(e) d_e(f^*(e)) + \frac{1}{4} f(e) d_e(f(e))$$

Proof. If $f^*(e) \geq f(e)$, then the result is immediate by the monotonicity of $d_e(\cdot)$. Now, consider the following diagram for when $f^*(e) < f(e)$:



Observe that the area a is exactly $f^*(e)d_e(f(e)) - f^*(e)d_e(f^*(e))$ and that the area of the rectangle bounded by $f(e)$, the axes, and the top-most dashed line is exactly $f(e)d_e(f(e))$. Then, Theorem 1 states that the area a is no greater than $\frac{1}{4}$ the area of the large rectangle. But, of course, this is the case as the a cannot have area greater than $\frac{1}{2}$ of the triangle enclosing it, and the triangle enclosing it cannot have area greater than $\frac{1}{2}$ of the total rectangle. ■

As promised, this yields an upper bound on the Price of Anarchy.

Corollary 1. *If $d_e(\cdot)$ is a linear function, then Price of Anarchy $\leq \frac{4}{3}$.*

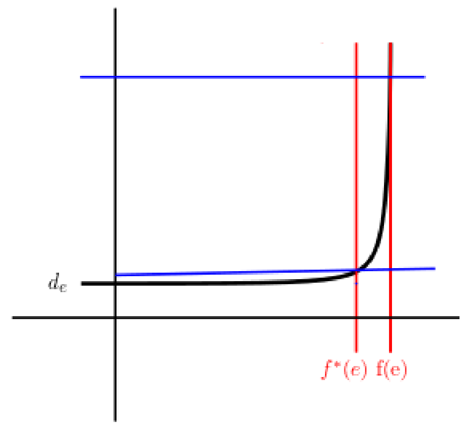
Proof. We have directly from applying Theorem 1 and Equation 3 that:

$$\begin{aligned} SC(f) &\leq \sum_e f^*(e)d_e(f(e)) \\ &\leq \sum_e \left[f^*(e)d_e(f^*(e)) + \frac{1}{4}f(e)d_e(f(e)) \right] \\ &= SC(f^*) + \frac{1}{4}SC(f) \end{aligned}$$

which directly implies the desired result. ■

1.2 Broader Delay Classes

We next consider a broader class of functions than linear delays and find that even for monotonically increasing, $\mathcal{C}^1$ (continuously differentiable), convex functions, we may find arbitrarily bad Price of Anarchy bounds. In particular, note that $\frac{\lambda}{1-\mu}$ can be made arbitrarily large as the type of construction in Figure §1.2 (appealing to the proof we have seen above to recover the values for λ, μ). A concrete example of such a pathological function would be x^n for sufficiently large n (and the figure below resembles such a function).
For completeness, we note that such a choice of pathological function can be directly translated to a pathological nonatomic flow problem instance by taking the Pigou graph with delay of 1 on one edge and delay on the edge specified by a pathological function as described above (with the rate being 1).



1.3 Modified Price of Anarchy bounds

Given that Price of Anarchy bounds (using the methods explored in this lecture) may be unattainable for broader classes, we consider a modification to the underlying structure for Price of Anarchy that permits us to find a (modified) Price of Anarchy bound. In particular, consider two nonatomic flow instances: one with rates r_i and one with rates $2r_i$ with all other parameters the same. Consequently, the latter problem requires satisfying twice the demand that is required for the former. Denote the Nash equilibrium for the former/easier problem to be f and `OPT` for the latter/harder problem to be f_2^*. We will provide a modified Price of Anarchy bound that relates the two (note that this is explicitly not an apples-to-apples comparison in some sense), that is we will find a constant k for all problem instances/games such that:

$$\frac{SC(f)}{SC(f_2^*)} \leq k \tag{5}$$

We first observe that an adjusted version of Equation 3 holds as is indicated below:

$$2SC(f) = 2\sum_e f(e)d_e(f(e)) \leq \sum_e f_2^*(e)d_e(f(e)) \tag{6}$$

Note that a magic inequality, as in Equation 4, trivially holds for $\lambda = 1, \mu = 1$ due to d_e being monotonically increasing. Consequently, this yields a (modified) Price of Anarchy bound of 1 as:

$$2SC(f) \leq \sum_e f_2^*(e)d_e(f(e)) \leq SC(f_2^*) + SC(f) \tag{7}$$

Intuitively, this indicates that for a nonatomic flow instance and its harder analogue, where the harder analogue is the identical instance with doubled rates, any Nash equilibrium in the original problem is at least as good of a solution as the optimal solution in the harder problem.