CS 6840: Algorithmic Game Theory	**Spring 2017**

Lecture 40: May 8

Lecturer: Éva Tardos *Scribe: Johan Bjorck*

Last time we studied an online scenario where potential buyers arrived sequentially and we offered them a potential price for a single item that they had to ability to refuse or accepts. The buyer's had valuations v_i drawn independently from distributions D_i, and by setting the price equal to

$$p = \frac{1}{2}\mathbb{E}[\max_i v_i]$$

the expected social welfare, the valuation of the player getting the item, was at least half of the expected maximum, the price we set. The problem was showing that the loss in utility when no player bought the item, was offset by the utility when a player that valued the item higher then its price bought it. The bound 1/2 is tight, however there are other methods for setting the price that acieves the bound.

We now study a more general setting, where we have multiple items $j = 1, 2, .., m$ and unit demand bidders with valuations $v_{ij} \sim F_i$, drawn from independent distributions (that is, buyer's valuation of different items can be correlated, but different buyers are independent). As before buyers arrive online, one-at-a-time. They see prices and are free to select their favorite item. We assume they will select the item $\max_{j\in R} v_{ij} - p_j$ of this maximum is nonnegative (or positive), where R is the set of items not yet bought by a previous person.

Theorem 40.1 *There exits prices p_j for the above setting determined by the distributions F_i, such that $\mathbb{E}[SW] \geq \frac{1}{2}\mathbb{E}[\max SW]$.*

Proof: For item j we set the price p_j equal to $\frac{1}{2}\mathbb{E}[\max SW_j]$, where SW_j is the utility generated in j in the optimal allocation, that is the $\mathbb{E}(v_{ij})$ for i is the person assigned item j in the allocation maximizing the sum of values. With the players arriving one at-a-time, let q_j denote the probability that item j is sold. We will estimate revenue and player utility as we did last time. The expected revenue is then

$$\sum_j q_j p_j$$

Consider player i. The utility for for player i has to be $\geq (v_{ij} - p_j)^+(1-q_j)$ for any item j, as with probability $(1 - q_j$ she will have a chance to buy item j and hence have utility $(v_{ij} - p_j)^+ = \max(v_{ij} - p_j, 0)$. This is true for all items, so player i's utility is at least $\max_j(v_{ij} - p_j)^+(1 - q_j)$. Now let us define the indicator variable x^*_{ij} that is 1 iff player i gets item j in the optimal allocation and 0 otherwise. Now the utility for a single person i has to be

$$\mathbb{E}(u_i) \geq \max_j(v_{ij} - p_j)^+(1 - q_j) \geq \mathbb{E}\left[\sum_j (v_{ij} - p_j)^+(1 - q_j)x^*_{ij}\right]$$

The expectation of the sum of revenue and utility for all players can then be expressed as

$$\begin{aligned}
\mathbb{E}\Big[SW\Big] &\geq \sum_j p_j q_j + \mathbb{E}\Big[\sum_{i,j}(v_{ij}-p_j)^+(1-q_j)x^*_{ij}\Big] \\
&\geq \sum_j p_j q_j + \mathbb{E}\Big[\sum_{i,j}(v_{ij}-p_j)(1-q_j)x^*_{ij}\Big] \\
&= \sum_j p_j q_j + \mathbb{E}\Big[\sum_{i,j} v_{ij}(1-q_j)x^*_{ij}\Big] - \mathbb{E}\sum_{i,j}\Big[p_j(1-q_j)x^*_{ij}\Big]
\end{aligned}$$

where we got the inequality and the next equation by removing the $()^+$ operator, and using linearity of expectation.

Now, using the fact that $\mathbb{E}\big[\sum_i v_{ij}x^*_{ij}\big] = \mathbb{E}[SW_j]$ and $\sum_i x^*_{ij} \leq 1$, we get

$$\geq \sum_j p_j q_j + \sum_j \Big(\mathbb{E}[SW_j](1-q_j) - p_j(1-q_j)\Big)$$

By definition of our price we have $2p_j = \mathbb{E}[SW_j]$, using this substitution we get

$$= \sum_j \Big(p_j q_j + 2p_j(1-q_j) - p_j(1-q_j)\Big) = \sum_j p_j = \Big(\frac{1}{2}\mathbb{E}[SW_j]\Big) = \frac{1}{2}\mathbb{E}[Opt]$$

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Finally, consider how to do this, if we have to learn valuations from data. Assume, you have some sample of valuations, and can compute an estimate of $\mathbb{E}(SW_j)$ from data. Of course the estimate will never be exactly correct. So it is important to know how sensitive this result is the setting of prices. We note that if we set the prices slightly lower or higher by some δ, we get a welfare estimate:

$$\begin{aligned}
\mathbb{E}[SW] &\geq \sum_j p_j q_j + \sum_j \Big(\mathbb{E}[SW_j](1-q_j) - p_j(1-q_j) \\
&\geq \sum_j \Big(p_j q_j + 2(p_j-\delta)(1-q_j) - p_j(1-q_j)\Big) = \sum_j p_j - 2\delta\sum_j\sum_j(1-p_j) \\
&= \Big(\frac{1}{2}\mathbb{E}[SW_j]\Big) - 2m\delta = \frac{1}{2}\mathbb{E}[Opt] - 2m\delta
\end{aligned}$$

where m is the number of items.

Finally, we note that in a multi-item setting estimating the SW_j for all j with small sample sizes seems hard. We note that for a single item case, we could make the proof work by sampling everyone's valuation once and then taking $p = \max_i v_i$. This different way to set the price also results in showing that social welfare is at least $1/2$ of the expected maximum value in expectation. I don't know if the analog for multiple item would work.