CS 6840: Algorithmic Game Theory **Spring 2017**

Lecture 30: April 14

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30.1 Data and Price of Anarchy in Routing Games

Given test data, it is easy to determine the actual delay and get the "optimal" routing. Recall, the optimal routing is the routing of existing traffic such that we minimize the sum of the delays, min $\sum$ delay. Further recall that for a given edge e with traffic x_e, and cost $c_e(x_e)$, the total cost along the edge is $x_e c_e(x_e)$. Therefore the "optimal" routing objective function is

$$\min \sum_e x_e c_e(x_e)$$

Our hope is that is that $x_e c_e(x_e)$ is a convex function, since, if so, convex optimization can be used to determine the optimal routing. This is equivalent to asking whether $(x_e c_e(x_e))' = x_e c_e'(x_e) + c_e(x_e)$ is monotonically increasing. In general these functions tend to be convex.

30.2 Empirical Price of Anarchy

All of our theorems so far are for finding upper bounds on the Price of Anarchy. However, we can also reason about Price of Anarchy empirically. The empirical cost of Nash is just $\frac{\text{cost Nash}}{\text{cost optimal}}$ evaluated on an example.

30.3 Data and Price of Anarchy in Auctions

The data we can get from routing games is different than what we can get in auctions. For example, consider an internet add auction, where player i has a value v_i for being clicked on. The data that we can get will contain list of auctions, winners, and payments, but no v_i's. Without v_i we cannot evaluate the social welfare of the auction and cannot find the optimal. So our goal for this section is to extract a v_i from the data.

30.3.1 Extracting values in Generalized Second Price auctions

The form of a GSP is as follows

- Multiple add slots 1,2,3, ... n
- Slot i has a click rate of α_i
- Person j has a relevance of γ_j.

- If person j gets slot i, $P(\text{click}) = \gamma_j * \alpha_i$
- Bids $b_1, b_2, ..., b_n$. Assume indexing such that $b_1\gamma_1 \geq b_2\gamma_2 \geq ... \geq b_n\gamma_n$
- Payment per click for player i can be computed as $p_i\gamma_i = b_{i+1}\gamma_{i+1} \implies p_i = \frac{b_{i+1}\gamma_{i+1}}{\gamma_i}$ (Assuming $b_1\gamma_1 \geq b_2\gamma_2 \geq ... \geq b_n\gamma_n$)

Let's consider a 1-shot full information game. In this game, every player knows all γ_j, α_i, and other player's bids. We now want to try and infer values from the bids.

We can get a bound on v_i's using the Nash condition. We know player j's utility in slot i is $\gamma_j\alpha_i(v_j - p_j)$, where the orice p_j that j pays per click depends on the slot. For ease of notation, let's sort players by $b_n\gamma_n$, so player i is in slot i. Then, player i's utility in slot i is just $\gamma_i\alpha_i(v_i - p_i)$.

Now consider what happens if player i deviates to a slot k. There are two possible cases for i's new utility

(a) $k > i$: $\gamma_i\alpha_k(v_i - p_i')$, where $p_i' = \frac{\gamma_{k+1}b_{k+1}}{\gamma_i}$, computed by the same formula used for slot k.

(b) $k < i$: $\gamma_i\alpha_k(v_i - \frac{\gamma_k b_k}{\gamma_i})$. The last term is no longer computed using b_{k+1} and player i is now also above k, so the original kth bidder becomes the $k+1$st in the new ordering.

Under the Nash condition, we know that for player i, being in slot i maximizes his utility. With this condition and the above utilities, we can generate $n-1$ inequalities of the form *utility slot* $k \leq$ *utility slot* i. Solving these inequalities will give us an upper and lower bound on v_i. Specifically, inequalities with $k > i$ considered in (a) will give us lower-bounds on v_i, as $\alpha_i\gamma_i > \alpha_k\gamma_i$, these lower slots offer less clicks at a cheaper price, which would be preferable if the value v_i was low. Inequalities with $k < i$ will give us upper-bounds on v_i, considered in (b) will give us upper-bounds on v_i, as $\alpha_i\gamma_i < \alpha_k\gamma_i$, these higher slots offer more clicks at a higher price, which would be preferable if the value v_i was high.

However, with $n-1$ inequalities (for n slots) this won't be able to pinpoint the exact value v_i. Another issue is that these inequalities relied on the assumption that the outcome of the game at this one time step is at equilibrium. It is reasonable that the players are well enough informed to have the equilibrium condition hold at every time step.