CS 6840: Algorithmic Game Theory **Spring 2017**

Lecture 29: April 12

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We have previously looked at Vickrey auctions, or second-priced auctions, and discovered that this auction mechanism is truthful. We will now generalize this to the broader Vickrey-Clarke-Groves (VCG) Mechanism.

29.1 Vickrey-Clarke-Groves (VCG) Mechanism

Suppose there are n players and a set of X outcomes. Player i has value $v_i(x)$ for each possible outcome $x \in X$. The goal of the mechanism is to select an outcome x^* that maximizes social welfare $\sum_i v_i(x)$

$$x^* = \operatorname*{argmax}_{x \in X} \sum_i v_i(x)$$

The payment scheme charges each player with their *damage* to society. The payment for player i is specified as

$$p_i = \max_x \sum_{j \neq i} v_j(x) - \sum_{j \neq i} v_j(x^*)$$

The optimal social welfare for other players when player i is not participating is $\max_x \sum_{j \neq i} v_j(x)$. The social welfare for other players in the selected outcome is $\sum_{j \neq i} v_j(x^*)$. Thus, each player pays how much his existence in the mechanism has hurt the others.

The utility received by player i is $v_i(x) - p_i$.

29.2 Truthfulness of VCG Mechanism

Theorem 29.1 *Reporting an agent's true value is a dominant strategy in the VCG mechanism.*

Proof: Let us look at player i's utility in this setting

$$\begin{aligned} u_i &= v_i(x^*) - p_i \\ &= v_i(x^*) - \left[\max_x \sum_{j \neq i} v_j(x) - \sum_{j \neq i} v_j(x^*)\right] \\ &= \sum_j v_j(x^*) - \max_x \sum_{j \neq i} v_j(x) \end{aligned}$$

By reporting a valuation different than the true valuation $v_i(.)$ the player can change the selected x^*. So its interesting to look at what selection x^* gives him maximum value. In the expression above, player i has no control over the second value $\max_x \sum_{j \neq i} v_j(x)$, so the choice of the mechanism is exactly aligned by the choice of player i; they both want to maximize the term $\sum_j v_j(x)$. Or phrased differently: by reporting a different value function $\bar{v}_i(.)$, player i would change the outcome to

$$\bar{x}^* = \operatorname*{argmax}_{x \in X} (\bar{v}_i(x) + \sum_{j \neq i} v_j(x))$$

and then have the utility

$$u_i = v_i(\bar{x}^*) - p_i = \sum_j v_j(\bar{x}^*) - \max_x \sum_{j \neq i} v_j(x) \leq \sum_j v_j(x^*) - \max_x \sum_{j \neq i} v_j(x)$$

by the choice of x^*, so this is maximized when he reports his true value $v_i(.)$.

We see here that the first value in the original utility function, $\sum_i v_i(x^*)$, is maximized for each player's true values. Therefore it is in the player's best interest to bid his actual value. ■

We also note that the payment is always nonnegative, as for the x maximizing the second term, $\max_x \sum_{j \neq i} v_j(x) \geq \sum_{j \neq i} v_j(x^*)$. And as long as $v_i(x) \geq 0$ for all x, the utility is nonnegative also, that is $p_i \leq v_i(x^*)$, this is true as $\max_x \sum_j v_j(x) \leq \sum_j v_j(x^*)$, so

$$u_i = v_i(x^*) - p_i = \sum_j v_i(x^*) - \sum_{j \neq i} v_j(x) \geq v_i(x) \geq 0$$

29.3 Issues with VCG Mechanism

The set of X can be impossibly large, as it covers every possible outcome in a given setting. Finding the optimal outcome x^* can then be very hard. Approximation optimization may not work well here either, or at least not result in truthful mechanisms. We have seen this at problem 4 on Problem set 3.

VCG is not budget-balanced. A mechanism resulting in total transfer payments adding up to zero may not exist. In fact, as you'll see on problem set 4, items may be sold for a total payment of 0!

The mechanism requires players to have data. A player must already have some knowledge on an outcome's impact in order to provide his perceived value. This is also an issue you will be on problem set 4 with the VCG used for ad-auctions.