CS 6840: Algorithmic Game Theory **Spring 2017**

Lecture 28: April 10

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28.1 Learning in Multi-Item Simultaneous Auctions

Suppose that there are n bidders and m items. Denote the set of all items by S. A bid b_{ij} for player i specifies an item and a bid for that item. In this setup there are uncountably many strategies available to a bidder. In order to apply the multiplicative weights algorithm, we need to discretize the possible bids that agents can report. For simplicity, suppose that $v_{ij} \in [0,1]$ for all i and j. Bids are then $b_{ij} \in \delta\mathbb{Z}$; i.e. agents may bid in increments of δ. How does discretizing bids affect regret? Should bids be rounded up or down to fit the discretization? The latter question has no clear-cut answer. Agents could round bids down and risk not receiving goods while paying a lower price, or agents could round up and pay a higher price for a better chance at a bundle. In particular, agents could receive negative payoffs by rounding their bids up (if $\delta > v_{ij}$). This is really an issue for coarse discretizations, so making δ sufficiently small should minimize this issue. In this lecture, we'll assume that agents round their bids up so that a reported bid is converted as: $b_{ij} \to \delta \left\lceil \frac{b_{ij}}{\delta} \right\rceil$

Rounding bids in this way can yield an additional regret of at most $m\delta$ for any given bidder (an extra regret of δ is possible for every item j). Despite solving the problem of uncountably many alternatives to bidders, this specification still yields two undesirable features.

- Running time per step: $(\frac{1}{\delta})^m$; e.g. $\delta \sim \frac{1}{m} \Rightarrow$ running time m^m per step
- Full information regret: $\frac{mlogm}{\epsilon}$

Notice that the running time per step can be quite large if there are many items and a fine discretization. In this lecture, we'll consider a less demanding bid structure.

Suppose that agents bid a single number b_i for all items. Players are ordered according to decreasing bids and sequentially choose their favorite items from what remain, paying price b_i for each item that they claim.

Under these conditions, the multiplicative weights algorithm has the following features:

- Running time per step: $\frac{1}{\delta}$
- Full information regret: $\frac{log\frac{1}{\delta}}{\epsilon}$
- Partial information regret: $\frac{1}{\epsilon\delta}$

These results will be useful in the next lecture.

28.2 Price of Anarchy Bounds

Consider the following simple case: bidder i has a valuation v_i for all items $j \in S_i \subseteq S$. This implies $v_i(S) = v_i|S \cap S_i|$.

Theorem 28.1 *Suppose agents submit a single bid for all items without rounding. Then the auction has a price of anarchy bound of 2 in the simple case above.*

Proof: The result follows from smoothness. In particular, we seek to show that the game is $(\frac{1}{2}, 1)$ smooth. Let b be a vector of bids for each player. Since this is a first-price setting, the optimal bid for agents is $b_i^* = \frac{v_i}{2}$. Let $S_i^* \subseteq S_i$ be the items that i receives in the optimum. We seek to show that

$$u_i(b_i^*, b_{-i}) \geq \frac{1}{2} v_i |S_i^*| - \sum_{j \in S_i^*} p_j(b)$$

When bidder i bids b_i^* on some item $j \in S_i^*$, there are two possible outcomes:

- i wins j with b_i^* and receives utility $\frac{v_i}{2}$
- i does not win item j, which implies that i gets utility 0 from item j and some other agent paid a higher price so that $p_j(b) \geq b_i^* = \frac{1}{2} v_i$

In the first case, the result holds with equality for good j before subtracting the price $p_j(b)$. In the second case, we have that the agent receives no utility from good j and $\frac{1}{2} v_i - p_j(b) \leq 0$. Summing over all goods, we have that

$$u_i(b_i^*, b_{-i}) \geq \frac{1}{2} v_i |S_i^*| - \sum_{j \in S_i^*} p_j(b)$$

Thus the game is $(\frac{1}{2}, 1)$ smooth, and so the price of anarchy bound is 2. ■

The above theorem showed a price of anarchy bound for a very simple specification of agent valuations. Now, suppose that set valuations are fractionally subadditive so that $v_i(S) = \max_k \sum_{j \in S} v_{ij}^k$. Recall that k in this setting denotes agent i's usage of good j. Allowing valuations to be fractionally subadditive raises the price of anarchy for this auction.

Theorem 28.2 *Suppose valuations are fractionally subadditive and that agents still bid a single value for all items they desire. Then the price of anarchy for this auction is bounded by $2H_m$, where H_m is the m^{th} harmonic number.*

Proof: Again, letting S_i^* be the set of goods that bidder i receives at the optimum, we have that

$$v_i(S_i^*) = \max_k \sum_{j \in S_i^*} v_{ij}^k = \sum_{j \in S_i^*} v_{ij}^{k_i}$$

so k_i is the additive valuation that determines the bidders value for her optimum bundle.

We want to find a possible bidding strategy b_i^* for i, so that not-regretting b_i^* gives us the price of anarchy bound. Define $v_{i\mu}^{k_i} = argmax\ v_{ij}^{k_i} |\{l \in S_i^* : v_{il}^{k_i} \geq v_{ij}^{k_i}\}|$. Let $b_i^* = \frac{v_{i\mu}^{k_i}}{2}$. When bidder i bids b_i^* in this auction, then for each item l such that $v_{il}^{k_i} \geq v_{i\mu}^{k_i}$, there are two possible cases:

- Bidder i has the option of taking item l with bid b_i^* and hence $v_{il}^{k_i} - b_i^* \geq b_i^*$.
- Bidder i does not have item l available, which implies a utility of 0 from good l and price $p_l(b) \geq b_i^*$.

Now consider the utility bid b_i^* gets against the bid vector b_{-i} of other bidders. Let $\tilde{S}_i^* = \{j : v_{ij}^{k_i} \geq v_{i\mu}^{k_i}\}$ and let $\tilde{S}_i \subset \tilde{S}_i^*$ denote the set of items still available in the auction when i with b_i^* gets to select items. Clearly, i can select this set, which would give him utility

$$v_i(\tilde{S}_i) - |\tilde{S}_i| b_i^* \geq \sum_{j \in \tilde{S}_i} (v_{ij}^{k_i} - b_i^*) \geq \sum_{j \in \tilde{S}_i^*} (b_i^* - p_j(b))$$

where the last inequality follows as for items in $j \in \tilde{S}_i$ we have $v_{ij}^{k_i} \geq 2b_i^*$, and for the remaining items in $\tilde{S}_i^*$ are no longer available, so the price is above b_i^*.

Now in the auction, i gets to select the subset of remaining items giving her maximum utility, so her utility is at least this big, so we get the bound

$$u_i(b_i^*, b_{-i}) \geq \sum_{j \in \tilde{S}_i^*} (b_i^* - p_j(b))$$

If we could show that $2b_i^* |\tilde{S}_i^*| = v_{i\mu}^{k_i} |\tilde{S}_i^*| \geq \frac{v_i(S_i^*)}{H_m}$ then we could continue this as

$$u_i(b_i^*, b_{-i}) \geq \frac{1}{2H_m} v_i(S_i^*) - \sum_{j \in \tilde{S}_i^*} p_j(b) \geq \frac{1}{2H_m} v_i(S_i^*) - \sum_{j \in S_i^*} p_j(b)$$

showing that the auction is $(\frac{1}{2H_m}, 1)$-smooth, and hance has a price of anarchy of at most $2H_m$ as claimed. We'll prove this claim as a separate lemma. ∎

To simplify the notation, we can drop i, and k_i from the notation. What we need to prove is the following.

Lemma 28.3 *Given a vector of values v_j, and a set S with $|S| \leq m$, define $\mu = argmax_j v_j |l \in S, v_l \geq v_j|$, then*

$$v_\mu |\{j \in S : v_j \geq v_\mu\}| \leq v(S) \leq H_m v_\mu |\{j \in S : v_j \geq v_\mu\}|$$

Proof: The first inequality true for any μ by definition. To see the second one, let $v_\mu |\{j \in S : v_j \geq v_\mu\}| = W$. Since W is the maximum $W = \max_j v_j |l \in S, v_l \geq v_j|$, the maximum value of v_j in the set S is at most W, the second highest can be at most $W/2$, the third highest at most $W/3$, and the t highest at most W/t, as there S can have at most $t-1$ elements with $v_j > W/t$ (or else the maximum would be higher). Using these bounds we get that

$$\sum_{j \in S} v_j \leq W + W/2 + W/3 + \ldots + W/m = H_m W$$

as claimed. ∎

Lecture based on paper Simple Mechanisms for Agents with Complements, Michal Feldman, Ophir Friedler, Jamie Morgenstern, Guy Reiner, that appeared in ACM EC'16, and is at `https://arxiv.org/abs/1603.07939`