CS 6840: Algorithmic Game Theory **Spring 2017**

Lecture 24: March 24

Lecturer: Éva Tardos *Scribe: Erin Yandel*

Valuation Classes

Up to now we have had single item or unit demand where the value of a set $v(S) = \max_{j \in S} v_j$. This value (as all the ones below) is what we normally call v_i (the value to one person).

24.1 Subadditive

If A, B are sets and $v(A)$ and $v(B)$ are the values of these sets then

$$v(A) + v(B) \geq v(A \cup B)$$

We assume this inequality always holds for this class, since without it it is difficult to do anything meaningful.

We will also assume that the value functions are normalized. So $v(\emptyset) = 0$ and $v(S) \leq v(S')$ if $S \subseteq S'$, i.e., there is free disposal. These two together also imply that $v(S) \geq 0$ for all S.

24.2 Decreasing Marginal Utility

If $S \subseteq S'$ and j is an item then

$$v(S + j) - v(S) \geq v(S' + j) - v(S')$$

Where $v(S + j) - v(S)$ is the marginal utility of item j when added to set S.

With the assumption that $v(\emptyset) = 0$, the subadditive inequality can be re-written in a form closer to this one:

$$v(A) - v(\emptyset) \geq v(A \cup B) - v(B)$$

Theorem 24.1 *Decreasing Marginal Utility* $\implies$ *Subadditive*

We propose that Decreasing Marginal Utility $\implies$ $\forall\ S \subseteq S'$ and A, $v(S \cup A) - v(S) \geq v(S' \cup A) - v(S')$. We prove this claim by induction on $|A|$:

Proof: Say $j \in A$, $A' = A \setminus \{j\}$ then by induction $v(S \cup A') - v(S) \geq v(S' \cup A') - v(S')$. By definition $S \cup A' \leq S' \cup A' \implies v(S \cup A' + j) - v(S \cup A') \geq v(S' \cup A' + j) - v(S' \cup A')$. Now since $A' + j = A$ we can see that this is the sum we wanted.

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Corollary 24.2 $S = \emptyset \implies$ *Subadditive inequality where* $B = S'$.

An alternative way of writing the Decreasing Marginal Utility inequality is

$$v(A) + v(B) \geq v(A \cap B) + v(A \cup B)$$

for all sets A and B.

In this form Decreasing Marginal Utility is called Submodular (means the same thing but used in different fields). **Proof:** Rearranging this we get

$$v(A) - v(A \cap B) \geq v(A \cup B) - v(B)$$

which is the same as the equation in the proof above if $S = A \cap B$ and $S' = B$.

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24.3 Fractionally Subadditive

Fractionally Subadditive is a version of Subadditive where you can take sets fractionally. So we now have a multiplier x_A, sets A, for all sets.

Set S is covered if $\sum_{A:i \in A} x_A \geq 1 \forall i \in S$.

If S is covered by x then $\sum_A x_A v(A) \geq v(S)$.

Theorem 24.3 *Fractionally Subadditive $\implies$ Subadditive since $x_A = x_B = 1$ makes the set $S = A \cup B$ covered.*

24.4 XOS

This valuation class is algorithmically nice to use but looks very different than the others.

An additive valuation is defined by having value $v_j \forall$ items j. The total value of a set S before was $v(S) = \sum_{j \in S} v_j$. Instead we now have multiple possible values for each item b_j^k, and use

$$v(S) = \max_k \sum_{j \in S} v_j^k$$

Given v_i^k for $k = 1, ..., n$ on items, where the k values represent that the item may have different values depending on its different uses.

Claim 24.4 *unit demand is a special case of XOS*

We have from earlier that unit demand uses $v(S) = \max_{j \in S} v_j$. This function has no k so we must make a k to fit the function. We use

$$v_j^k = \begin{cases} v_j & j = k \\ 0 & otherwise \end{cases}$$

So that we have the vector $\vec{v_j} = [0, ..., v_j, 0, ..., 0]$.

Claim 24.5 *XOS is Subadditive*

Proof:

We define

$$v(A \cup B) = \max_k \sum_{j \in A \cup B} x_j^k = \sum_{j \in A \cup B} v_j^{k^*}$$

that is, let k^* be the value where the maximum occurs for the set $A \cup B$. Now we have

$$v(A \cup B) = \sum_{j \in A \cup B} v_j^{k^*} \leq \sum_{j \in A} v_j^{k^*} + \sum_{j \in AB} v_j^{k^*} \leq \max_k \sum_{j \in A} x_j^k + \max_k \sum_{j \in B} x_j^k$$

where the first inequality is true as the items in $A \cap B$ are now included twice, and the second inequality is true as k^* is one possible value for the k in the max.

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Claim 24.6 *XOS is Fractionally Subadditive.*

Proof:

Same as Subadditive proof above but now we have x_A

We have x_A sets and S is covered. So $v(S) = \sum_{j \in S} v_j^{k^*}$, as before k^* is where the max occurs for set S. Using this and other equations from above we get that

$$\sum_A x_A v(A) = \sum_A x_A [\max_k \sum_{j \in A} v_j^k] \geq \sum_A x_A \sum_{j \in A} v_j^{k^*} = \sum_j v_j^{k^*} (\sum_{A : j \in A} x_A) \geq \sum_{j \in S} v_j^{k^*} = v(S)$$

where the last inequality is true because S is covered so for $hj \in S$ we have $\sum_{A:j \in A} x_A \geq 1$.

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Facts:

Fractionally Subadditive=XOS
Submodular $\implies$ XOS

(Proofs may or may not be covered in a different lecture).