CS 6840: Algorithmic Game Theory **Spring 2017**

Lecture 20: March 13

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20.1 Logistics

- Make sure to pair your partner before the deadline
- Next two problem sets will be easier or have fewer problems, so you can all work on the project also.
- Project proposal due Friday, March 17th
- Problem Set 2 will be graded by March 20th
- Problem Set 3 due before break, March 30

20.2 Recall multi-item auctions

Last lecture we considered auction with multiple items and unit demand. Let v_{ij} be the value of item j to buyer i. For any set of items S, let $v_i(S) = \max_{j \in S} v_{ij}$. Recall the bid $b_i^*(\mathbf{v})$ used in PoA proof: for all value vectors $\mathbf{v}$ and bid vectors $\mathbf{b}$,

$$\sum_i u_i(b_i^*(\mathbf{v}), \mathbf{b}_{-i}) \geq \frac{1}{2} Opt - rev(\mathbf{b}) \tag{20.1}$$

where the bid $b_i^*(\mathbf{v})$ is defined in following way: if item j is assigned to player i in $Opt(\mathbf{v})$, player i will bid $v_{ij}/2$ on item i.

20.3 PoA bound for Bayes-Nash equilibria

Today we consider a Bayesian case where the value of each player i are from distributions F_i, i.e., $v_{ij} \sim F_i$. In vector form we have $\mathbf{v} \sim \mathcal{F} = (F_1, F_2, \cdots, F_n)$. We assume each player's F_i are independent. Our goal today is to prove $PoA \geq 1/2$ for this Bayesian case.

20.3.1 Strategy to design deviation b_i^*

In this case, player i only knows the distributions of other players' value, so the previous deviation b_i^* is no longer well defined. The strategy to designing b_i^* in this case is following: for each player i, she samples $\mathbf{v}'_{-i}$ from distributions $\mathbf{F}_{-i}$, and bid $b_i^*(v_i, \mathbf{v}'_{-i})$ defined previously (when we know all $\mathbf{v}$).

20.3.2 PoA bound Proof

Let b be the Bayes-Nash. When $\mathbf{v}$ is selected from the distribution F, the resulting bid vectors follow a distribution $\mathcal{G} = (g_1, g_2, \ldots, g_n)$. Note that the distributions g_i are independent, and also g_i is independent of v_j for any $j \neq i$.

Now for all i and fixed v_i,

$$\begin{aligned} \mathbb{E}_{\mathbf{v}\sim\mathcal{F},\mathbf{b}=b(\mathbf{v})}[u_i(\mathbf{b}, v_i)] &\geq \mathbb{E}_{\mathbf{v}'_{-i}\sim\mathbf{F}_{-i},\mathbf{v}_{-i}\sim\mathbf{F}_{-i},b_{-i}=b(v_{-i}}[u_i(b_i^*(v_i, \mathbf{v}'_{-i}), \mathbf{b}_{-i}, v_i)] \\ &= \mathbb{E}_{\mathbf{v}'_{-i}\sim\mathbf{F}_{-i},\mathbf{b}_{-i}\sim\mathbf{g}_{-i}}[u_i(b_i^*(v_i, \mathbf{v}'_{-i}), \mathbf{b}_{-i}, v_i)] \end{aligned}$$

where the equation notes that the values v_{-i} were not relevant, so we dropped them in taking the expectation. Next, we change the notation v' to v (as v is used) on the RHS and sum over i we have

$$\mathbb{E}_{\mathbf{v}\sim\mathcal{F},\mathbf{b}=b(\mathbf{v})}[\sum_i u_i(\mathbf{b}, v_i)] \geq \mathbb{E}_{\mathbf{v}\sim\mathcal{F},\mathbf{b}\sim\mathcal{G},\text{indep}}[\sum_i u_i(b_i^*(v_i, \mathbf{v}_{-i}), \mathbf{b}_{-i}, v_i)]$$

where we note that the $\mathbf{v} \sim \mathcal{F}$ and $\mathbf{b} \sim \mathcal{G}$ is pulled independently, that is b is not the bid associated with the value vector v. Use the inequality (20.1) from the full information case and take expectation over $\mathbf{v} \sim \mathcal{F}$ and $\mathbf{b} \sim \mathcal{G}$,

$$\begin{aligned} \mathbb{E}_{\mathbf{v}\sim\mathcal{F},\mathbf{b}\sim\mathcal{G},\text{indep}}[\sum_i u_i(b_i^*(v_i, \mathbf{v}_{-i}), \mathbf{b}_{-i}, v_i)] \geq &\mathbb{E}_{\mathbf{v}\sim\mathcal{F},\mathbf{b}\sim\mathcal{G}}[\frac{1}{2}Opt(\mathbf{v}) - rev(\mathbf{b})] \\ = &\mathbb{E}_{\mathbf{v}\sim\mathcal{F}}[\frac{1}{2}Opt(\mathbf{v})] - E_{\mathbf{b}\sim\mathcal{G}}[rev(b)] \end{aligned}$$

Now move $E_{\mathbf{b}\sim\mathcal{G}}[rev(b)]$ to the LHS we show that the expected welfare of the Bayes-Nash equilibrium is at least $\frac{1}{2}$ times the expected maximum welfare, i.e., $PoA \geq \frac{1}{2}$.

20.3.3 General Smooth Mechanisms

If there exists bid $b_i^*(\mathbf{v})$ for each player i, depending on the values of all other players $\mathbf{v}$, such that for all bid profiles $\mathbf{b}$

$$\sum_i u_i(b_i^*, \mathbf{b}_{-i}) \geq \lambda \cdot Opt(1, 2, \cdots, n) - \mu \cdot rev(\mathbf{b})$$

Since $\mu = 1$ is a typical case, for full information auction in this case, we have

$$\mathbb{E}[SW(Nash)] \geq \lambda \cdot Opt$$

By the proof we just did today, for Bayes-Nash we have

$$\mathbb{E}[SW(Nash)] \geq \lambda \cdot \mathbb{E}_{\mathbf{v}}[Opt(\mathbf{v})]$$

20.4 Nash in 2nd price auction

Recall in single item 2nd price auction where bidders have value $v_1, v_2, \cdots, v_n$, we claimed that bidding the true value, i.e., $b_i(v_i) = v_i$ is a Nash. It is also dominant strategy, i.e.,

$$u_i(v_i, \mathbf{b}_{-i}) \geq u_i(b_i, \mathbf{b}_{-i})$$

for all b_i. This is true.

However, it is not true that every bidder should bid below their true value to be Nash. Here is an example:

Assume 2 bidders with value $v_1 = 99, v_2 = 2$ but their bids are $b_1 = 1, b_2 = 100$. We can check this is a Nash but clearly bidder 2 bids much higher than her true value.