CS 6840: Algorithmic Game Theory **Spring 2017**

Lecture 18: March 8

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18.1 Single-Parameter Mechanisms and Myerson's Lemma

18.1.1 Model and Definitions

In our auction, we have n players where player i has private value $v_i \in \mathbb{R}$. Each v_i is drawn from a distribution with cdf F_i. The v_i's are independent but not necessarily identically distributed.

Definition 18.1 (Mechanism) *A* mechanism (x, p) *is defined by an allocation rule and a payment rule. The allocation rule* $x : \mathbb{R}^n \to \{0, 1\}^n$ *maps a bid vector* $\vec{b}$ *to an allocation. The payment rule* $p : \mathbb{R}^n \to \mathbb{R}^n$ *maps* $\vec{b}$ *to the amount that each player must pay.*

We will write $x_i(\vec{b})$ and $p_i(\vec{b})$ for the ith coordinate of $x(\vec{b})$ and $p(\vec{b})$ respectively.

Example: For a first-price auction, the allocation rule is

$$x_i(\vec{b}) = \begin{cases} 1 & b_i = \max_j b_j \\ 0 & \text{otherwise} \end{cases}$$

Note that this requires a tie-breaking method to make sure that only one entry of $x(\vec{b})$ is 1. The payment rule is

$$p_i(\vec{b}) = b_i \cdot x_i(\vec{b}).$$

Assumptions: We assume that bidders are risk-neutral quasi-linear utitliy maximizers:

- *Risk-neutral*: The quality of a random outcome is evaluated as the expected quality of the outcome.
- *Quasi-linear*: The utility that a player gets is the value of the good she receives minus the payment she must make.

Putting these together, This means

$$u_i(v_i, \vec{b}) = \mathbb{E}[v_i \cdot x_i(\vec{b}) - p_i(\vec{b})].$$

18.1.2 Interim Allocations and Payments

n-player auctions are complex; our goal is to decompose the auction into single-player scenarios so we can consider each player on its own. Our device to decouple the players is to use interm allocations and payment rules.

Definition 18.2 (Interim Allocations and Payments) *For a mechanism (x, p), bid profile $\vec{b}$, and player i, the interim allocation rule is*

$$\overline{x}_i(v_i) = \mathbb{E}_{\vec{b}_{-i}}[x_i(b_i(v_i), \vec{b}_{-i})],$$

and the interim payment rule is

$$\overline{p}_i(v_i) = \mathbb{E}_{\vec{b}_{-i}}[p_i(b_i(v_i), \vec{b}_{-i})],$$

where $b_i(v_i)$ is the bid that player i makes when her value is v_i.

Note that $\overline{x}_i(v_i)$ and $\overline{p}_i(v_i)$ are just numbers, making them much easier to work with.

Example: Consider a second-price auction with 2 players who are playing truthfully, i.e. $b_1(\cdot)$ and $b_2(\cdot)$ are the identity functions. Let $v_1, v_2 \sim U([0,1])$. Then, the interim allocation rule is

$$\overline{x}_i(v) = v,$$

since if a player bids v, the probability that she wins is the probability that the other player's value is below v, which is exactly v. This means that the interim payment rule is

$$\overline{p}_i(v) = \left(\frac{v}{2}\right) v = \frac{1}{2}v^2$$

because player i wins with probability v, and her payment conditioned on her winning is equal to her opponent's expected value conditioned on being no more than v, which is $v/2$.

Key idea: When $\vec{b}$ is in equilibrium, $\overline{p}_i$ is uniquely determined by $\overline{x}_i$. We will show why this is true and use this to argue Myerson's Lemma for revenue equivalence.

Example: Suppose $\overline{x}_i(v)$ is given by

$$\overline{x}_i(v) = \begin{cases} 1 & v \geq 100 \\ 0 & \text{otherwise} \end{cases}$$

In equilibrium, it must be the case that

$$\overline{p}_i(v) = \begin{cases} 100 & v \geq 100 \\ 0 & \text{otherwise} \end{cases}$$

assuming that prices are normalized such that $\overline{p}_i(0) = 0$. Figure 18.1 shows $\overline{x}_i(v)$ and $\overline{p}_i(v)$ for this example. To see why this must be the case in equilibrium, observe that if $v < 100$, then $\overline{p}_i(v)$ must be 0, because if it were anything larger than 0, player i would have an incentive to lie and bid as if her value were 0, since the allocation would be unchanged, but the price she pays would decrease. Next, consider the case where $v = 100$. Since expected utility at $v' = 100 - \varepsilon$ is 0, expected utility at $v = 100$ must be 0; if it were any more, a player with utility $100 - \varepsilon$ would bid as if she has valuation 100, and if it were any less, a player with utility 100 would bid as if she has utility $100 - \varepsilon$. This means $\overline{p}_i(100) = 100$. Finally, for $v > 100$, it must be the case that $\overline{p}_i(v) = 100$, because otherwise player i would bid as if she had value 100.

Figure 18.2 shows a slightly more complex version of this example. Note that in both of these, for any value v, $\overline{p}_i(v)$ is the area above the $\overline{x}_i(\cdot)$ curve, between 0 and v on the horizontal axis and between the $\overline{x}_i(\cdot)$ curve and $\overline{x}_i(v)$. In fact, this is Myerson's Lemma:

Lemma 18.3 (Myerson's Lemma) *$\vec{b}$ is an equilibrium of (x, p) if and only if the interim allocation and payment rules satisfy*

- *$\overline{x}_i(v)$ is monotone nondecreasing in v.*

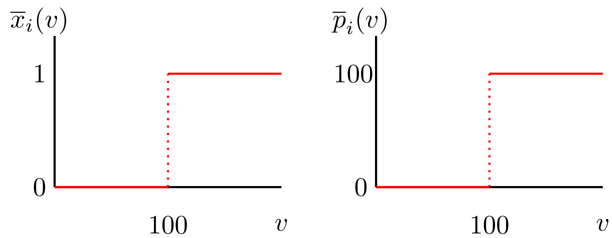

Figure 18.1: An example $\overline{x}_i(v)$ and $\overline{p}_i(v)$.

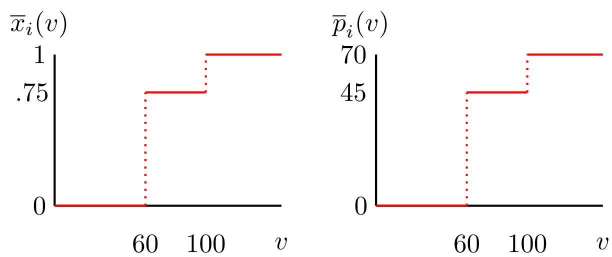

Figure 18.2: Another example $\overline{x}_i(v)$ and $\overline{p}_i(v)$.

- *For all i and v,*
$$\overline{p}_i(v) - \overline{p}_i(0) = \int_0^v \overline{x}_i(v) - \overline{x}_i(t)\, dt.$$

For some intuition as to why these conditions imply that $\vec{b}$ is an equilibrium, consider Figure 18.3. Player i's utility at v^* is
$$u_i(b_i(v^*), \vec{b}_{-i}) = v^* \cdot \overline{x}_i(v^*) - \overline{p}_i(v^*),$$
which, according to the conditions in Myerson's Lemma, is exactly the red shaded area under the $\overline{x}_i(v)$ curve. If player i overbids as if her value were v', her utility would decrease by the area of the green shaded region. If she instead underbids as if her value were v'', her utility would decrease by the area of the blue shaded region. As a result, the strategy is in equilibrium.

Myerson's Lemma implies that if any two mechanisms have the same expected allocation rule, then they have the same expected payments, which means the revenues the achieve are equivalent.

Going back to the auctions we've already discussed (first price, second price, all pay), we can see that in symmetric equilibria of these auctions, the player with the highest value always wins. As a result, these

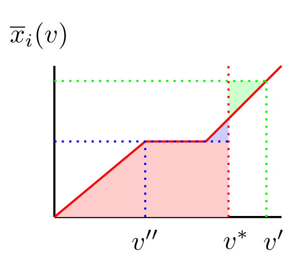

Figure 18.3: Intuition for Myerson's Lemma

auctions all have the same interim allocation rule, so they have the same expected payments. Previously, we found that for the second price auction, $\overline{p}_i(v) = v^2/2$, meaning that this is also true for the all pay auction. However, since the expected payment for the all pay auction is just the bid, it must be the case that at symmetric equilibrium, $b_i(v) = v^2/2$ in the all pay auction. This example shows how we can use Myerson's Lemma to derive simple characterizations of equilibria without too much work.