CS 6840: Algorithmic Game Theory	Spring 2017
Lecture 16: March 3	
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16.1 Admin

Description for the final project is now posted on CMS.

16.2 Overview

Recall from last lecture that, in a *simple (or first-price single-item) auction*,

- there are n players (or bidders or buyers)
- each bidder i has value v_i for the item, or maximum she is willing to pay. For a *full-information game*, $\boldsymbol{v} = (v_1, \dots, v_n)$ is a constant vector and is known to all players. For a *Bayesian (or incomplete-information game)*, each v_i is drawn from a distribution $\mathcal{F}_i$, but only the vector of distributions, $\boldsymbol{\mathcal{F}} = (\mathcal{F}_1, \dots, \mathcal{F}_n)$ is known to all players. Or possibly, the vector of distributions $\boldsymbol{\mathcal{F}}$ is a joint distributions with player values correlated.
- the strategy is then a vector of bids of the players $\boldsymbol{b} = (b_1, \dots, b_n)$ (in the Bayesian version, b_i is a function of v_i, or $\boldsymbol{b}(\boldsymbol{v})$)
- the winner of the game is player $\arg\max_i b_i$, with some arbitrary tie-breaking procedure
- the utility of each player is

$$u_i(\boldsymbol{b}; v_i) = \begin{cases} v_i - b_i & \text{if player } i \text{ wins} \\ 0 & \text{if player } i \text{ does not win} \end{cases}$$

Last week, we computed the Nash equilibria for

- the full information game
- the Bayesian game with two players where $\mathcal{F}_1 = \mathcal{F}_2 = \text{Unif}([0,1])$, and is drawn independently.

Today, we will show that the price of anarchy for an arbitrary Bayesian simple auction is at least 1/2. But to get started we will ignore the fact that we know the Nash equilibrium of the full-information game, and outline the proof for this easy case, generalizing it to the Bayesian game after. We will do this by considering for each player, what would happen if this player shades his/her bid to half their value.

16.3 Price of Anarchy for full-information games

Before getting to the bound, let's recall the definition of Nash for this game and define the social welfare for an auction.

A bid (or strategy) profile $\boldsymbol{b}$ is *Nash* if and only if for all players i and all alternative bids b_i',

$$u_i(\boldsymbol{b}; v_i) \geq u_i(b_i', \boldsymbol{b}_{-i}; v_i).$$

For the social welfare of this game, we consider the seller a player of the auction, whose utility is the revenue of the auction. Define $\text{rev}(\boldsymbol{b}) = \max_i b_i$. Then define the *social welfare* of a bid profile $\boldsymbol{b}$ is

$$SW(\boldsymbol{b}; \boldsymbol{v}) = \text{rev}(\boldsymbol{b}) + \sum_i u_i(\boldsymbol{b}; v_i).$$

This simplifies to $SW(\boldsymbol{b}; \boldsymbol{v}) = v_{i_0} - b_{i_0} + \text{rev}(\boldsymbol{b}) = v_{i_0}$, where player i_0 is the winner under $\boldsymbol{b}$. So the optimum is $\textsc{Opt}(\boldsymbol{v}) = \max_i v_i$ (*). Then the *price of anarchy* of an auction is

$$\min_{\boldsymbol{b} \text{ Nash}} \frac{SW(\boldsymbol{b}; \boldsymbol{v})}{\textsc{Opt}(\boldsymbol{v})}$$

Consider the case that player i bids half of their value; let $b_i^* = v_i/2$.

Lemma 16.1 *For all* $1 \leq i \leq n$, $u_i(b_i^*, \boldsymbol{b}_{-i}; v_i) \geq \max(v_i/2 - \text{rev}(\boldsymbol{b}), 0)$

Proof: Suppose player i wins, then she wins with bid $v_i/2$ and $u_i(b_i^*, \boldsymbol{b}_{-i}; v_i) = v_i - v_i/2 = v_i/2$. Moreover, if she wins her utility is nonnegative, so the inequality holds.

Suppose play i doesn't win. Then her utility is zero and the winner had a bid above $v_i/2$, so $v_i/2 - \text{rev}(\boldsymbol{b}) \leq 0$. So the bound holds.

■

Proposition 16.2 *The price of anarchy of a full information simple auction is at least* $1/2$.

Proof: Let $\boldsymbol{b}$ a Nash equilibrium, then let player i_0 (different i_0 to before) be such that $v_{i_0} = \max_i v_i$.

$$\begin{aligned}
SW(\boldsymbol{b}; \boldsymbol{v}) &= \text{rev}(\boldsymbol{b}) + \sum_{i=1}^n u_i(\boldsymbol{b}; \boldsymbol{v}) \\
&\geq \text{rev}(\boldsymbol{b}) + \sum_{i=1}^n u_i(b_i^*, \boldsymbol{b}_{-i}; v_i) && (\ \boldsymbol{b} \text{ is Nash}) \\
&\geq \text{rev}(\boldsymbol{b}) + v_{i_0}/2 - \text{rev}(\boldsymbol{b}) + \sum_{i \neq i_0} 0 && (\text{use 1st arg in 16.1 for } i_0 \text{ \& 2nd for all others}) \\
&= v_{i_0}/2 \\
&= \textsc{Opt}(\boldsymbol{v})/2 && (\text{by } (*)\ )
\end{aligned}$$

Since this holds for all Nash equilibria, PoA $\geq 1/2$. ■

16.4 Price of Anarchy for Bayesian game

Now we want to show that the same thing work in the Bayesian case if we carefully put expectations everywhere. To deal with the fact that $\boldsymbol{v}$ and $\boldsymbol{b}$ are random variables, we define two new random variables

$$x_i(\boldsymbol{b}) = \begin{cases} 1 & \text{if player } i \text{ wins with bid } \boldsymbol{b} \\ 0 & \text{otherwise} \end{cases}$$

$$x_i^*(\boldsymbol{v}) = \begin{cases} 1 & \text{if player } i \text{ has the highest value, with ties broken by a fixed rule} \\ 0 & \text{otherwise} \end{cases}$$

Here, we are leaning on the fact that the $\mathcal{F}_i$'s are real-valued continous random variables, so two selecting the same real number has probability 0. Otherwise, break ties some other arbitrary way.

A bid (or strategy) profile $\boldsymbol{b}$ is *Nash* if and only if for all players i and all alternative bids b'_i,

$$\mathbb{E}_{\boldsymbol{v}_{-i}}[u_i(\boldsymbol{b}; v_i)|v_i] \geq \mathbb{E}_{\boldsymbol{v}_{-i}}[u_i(b'_i, \boldsymbol{b}_{-i}; v_i)|v_i].$$

Same reasoning, different notation. The social welfare of a game with bids $\boldsymbol{b}(\boldsymbol{v})$ is the bid of the winner; optimum of a valuation $\boldsymbol{v}$ is the max value. With our new notation, that is

$$SW(\boldsymbol{b}; \boldsymbol{v}) = \sum_{i=1}^{n} v_i \, x_i(\boldsymbol{b}) \tag{16.1}$$

$$\textsc{Opt}(\boldsymbol{v}) = \sum_{i=1}^{n} v_i \, x_i^*(\boldsymbol{v}) \tag{16.2}$$

Then the price of anarchy of a distribution $\boldsymbol{\mathcal{F}}$ is the smallest

$$\frac{\mathbb{E}_{\boldsymbol{v}}[SW(\boldsymbol{b}(\boldsymbol{v}); \boldsymbol{v})]}{\mathbb{E}_{\boldsymbol{v}}[\textsc{Opt}(\boldsymbol{v})]}$$

over all Nash equilibria $\boldsymbol{b}$. Restating the lemma from the previous section using our new notation, we get

Lemma 16.3 *For all* $1 \leq i \leq n$*,*

$$u_i(b_i^*, \boldsymbol{b}_{-i}; v_i) \geq (v_i/2 - \text{rev}(\boldsymbol{b}))x_i^*(\boldsymbol{v})$$

Proposition 16.4 *The price of anarchy of a Bayesian simple auction is at least* $1/2$ *(even is player values are arbitrarily correlated).*

Proof: Let $\boldsymbol{b}$ a Nash equilibrium, so for all i,

$$\mathbb{E}_{\boldsymbol{v}_{-i}}[u_i(\boldsymbol{b}; v_i)|v_i] \geq \mathbb{E}_{\boldsymbol{v}_{-i}}[u_i(b_i^*, \boldsymbol{b}_{-i}; v_i)].$$

Take expectation over v_i on both sides to get

$$\mathbb{E}_{\boldsymbol{v}}[u_i(\boldsymbol{b}; v_i)] \geq \mathbb{E}_{\boldsymbol{v}}[u_i(b_i^*, \boldsymbol{b}_{-i}; v_i)]. \tag{16.3}$$

Then same thing as before,

$$\begin{aligned}
SW(\boldsymbol{b}; \boldsymbol{v}) &= \mathbb{E}_{\boldsymbol{v}}\left[\text{rev}(\boldsymbol{b}) + \sum_{i=1}^{n} u_i(\boldsymbol{b}; \boldsymbol{v})\right] \\
&\geq \mathbb{E}_{\boldsymbol{v}}[\text{rev}(\boldsymbol{b})] + \mathbb{E}_{\boldsymbol{v}}\left[\sum_{i=0}^{n} u_i(b_i^*, \boldsymbol{b}_{-i}; v_i)\right] && \text{(Equation 16.3)} \\
&= \mathbb{E}_{\boldsymbol{v}}[\text{rev}(\boldsymbol{b})] + \mathbb{E}_{\boldsymbol{v}}\left[\sum_{i=1}^{n} (v_i/2 - \text{rev}(\boldsymbol{b}))x^*(\boldsymbol{v})\right] && \text{(Lemma 16.3)} \\
&= \mathbb{E}_{\boldsymbol{v}}[\text{rev}(\boldsymbol{b})] + \mathbb{E}_{\boldsymbol{v}}[\textsc{Opt}(\boldsymbol{v})/2] - \mathbb{E}_{\boldsymbol{v}}\left[\text{rev}(\boldsymbol{b}) \sum_{i=1}^{n} x_i^*(\boldsymbol{v})\right] && \text{(Equation 16.2)} \\
&= \mathbb{E}_{\boldsymbol{v}}[\text{rev}(\boldsymbol{b})] + \mathbb{E}_{\boldsymbol{v}}[\textsc{Opt}(\boldsymbol{v})/2] - \mathbb{E}_{\boldsymbol{v}}[\text{rev}(\boldsymbol{b})] \\
&= \mathbb{E}_{\boldsymbol{v}}[\textsc{Opt}(\boldsymbol{v})]/2
\end{aligned}$$

Since this holds for all Nash equilibria, PoA $\geq 1/2$. ■

Note: This is not the best bound possible. If we let b_i^* be random, distributed on $[0,(1-1/e)v_i]$ with density function $f_i(x) = \frac{1}{v_i - x}$, we get a lower bound of $1 - 1/e \approx 0.63$ for the PoA [Syrgkanis and Tardos (2013)].

For those who have not seen/used density functions, this means that the probability of player i selecting a bid b that is at most x, when deviating to b_i^*, is $Pr(b \leq x) = F_i(x) = \int_0^x f_i(x)dx$. For this to be a distribution, we need that $f_i(x)$ is nonnegative (which it is), and $F_i((1-1/e)v_i) = 1$, which is indeed the case, as $\int f_i(x)dx = -\ln(v_i - x)$, so

$$\int_0^{(1-1/e)v_i} f(x)dx = -\ln(v_i/e) + \ln v_i = \ln e = 1.$$