Lecture 15: March 1

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15.1 Logistics

- Problem set 2 due March 8th
- Problem set 3 most likely due March 29th
- Project proposal will be due some time between problem set 2 and 3

15.2 Auctions

Up until now we have primarily been discussing congestion games, and more generally games in which players minimize cost. Starting today we will talk about auction games. An example of this is the federal government holding an auction to buy spectrum from television providers and sell it to cell phone providers. This is a two sided multi item auction, although it only occurs once so it is impossible to learn. We will focus on repeated games, with many buyers, sellers, and items, so that we can apply learning algorithms.

Today, we will start with a simple model of single item auctions.
An auction will be defined by:

- One item, n players (buyers)
- Each buyer i has a value v_i that they place on the item
- If a player buys the item at a price p, the utility they gain is $v_i - p$
- If a player does not buy the item, they have a utility of zero

The other thing to consider is the mechanism by which the auction runs. One example of this is Vickrey's auction, or 2^{nd} price auctioning. In this example, everyone gives their v_i value, and the maximum v_i is the winner of the auction. The price they pay for the item is the second highest v_i. This auction mechanism is notable because a consequence of it is that reporting your true v_i value is always a dominant strategy.This is part of a class of mechanisms which entail true reporting, also referred to as "Truthful mechanisms", unfortunately, with motre complex situation, this usually requires a rather complex mechanism.

In this class we will focus on simple mechanisms. Reporting your true value is often either not possible or not a good strategy. Thus we will first examine a single item game that allows for false reporting.

We can consider a game like above where there is one item, n players, and values $v_1 \geq v_2 \geq ... \geq v_n$, and all players have full information about the other players' values. In this game, player 1 always wins because their value of the item is the highest, and they should always bid ϵ more than the second highest, v_2. This is a Nash equilibrium if bidding is discretized or ties are broken in a specific way, but ultimately this is an unrealistic scenario because perfect information never occurs in life. Thus, we will discuss a more realistic version, the Bayesian version.

Bayesian version:

- We have one item, n players, and random $V_1 \geq V_2 \geq ... \geq V_n$ such that V_i is the value of the item for player i, each V_i is taken from a distribution F_i, and opponents know each others' distributions
- We will simplify this case by letting all the distributions be the same, so $(V_1, V_2, ..., V_n)$ are all taken from the same distribution F
- For a simple example of this, let F be the uniform distribution on $[0, 1]$, let $n = 2$, and let the players' values be mutually independent
- Each player i will have a function $b_i(v)$ that tells them what to bid if their value is v
- We will assume the players act symmetrically, so $b_1(v) = b_2(v) = b(v)$, though this may also be loss of generality
- Let us also assume b is monotone. With all these assumptions, we can solve for Nash

Now if player i is given their V_i to be some v, how do we solve for the optimal bid $b(v)$? First, assume you bid some $b(x)$. Then we know that $b(x)$ has to give a worse utility than $b(v)$ if $b(v)$ is the Nash equilibrium.
The utility of bidding $b(x)$ is $v - b(x)$ if you win, and zero if you lose.
The probability of winning is the probability that $b(x) > b(V_2)$.
Since b is monotonic, we can see that this is the probability that $x > V_2$.
On a uniform $[0, 1]$ distribution, this probability is x. Thus, the expected utility of bidding $b(x)$ with value v is $E(u(v, x)) = x(v - b(x))$
Now we say that since $b(v)$ is defined as the Nash equilibrium, this must be maximized at $x = v$. Thus, we maximize $x(v - b(x))$ by taking the derivative with respect to x and setting it equal to zero.

$v - (xb(x))' = 0$
$v = (xb(x))'$

Again, we know this is maximized at $x = v$, so we can use this to solve for b.

$v = (vb(v))'$
Integrate with respect to v
$\frac{v^2}{2} = vb(v)$
$b(v) = \frac{v}{2}$

Thus we have solved for the Nash equilibrium function $b(v)$. We need to check that the function $b(v)$ we found is monotone (as our argument assumed this), and also need to check that bidding a value that is not of the form of $b(x)$ is not a good strategy for any value v. This latter fact is true, as a value not of the form $b(x)$ is any number above 1/2, but bidding 1/2 es enough for winning the item 100% of the time, and hence any bigger bid is dominated by bidding 1/2.

We did this for a super simple case, symmetric bidding, $[0, 1]$ range for all, only two bidders. We can still solve this if we eliminate the $[0, 1]$ distribution range, or increase the number of players. However, if we eliminate symmetry of behavior, symmetry of range, or independence of values, it becomes extremely difficult to solve. Since we want to examine non-symmetric cases, we will instead consider proving something about equilibrium, or about something that is close to the equilibrium without actually finding the equilibrium.