CS 6840: Algorithmic Game Theory	**Spring 2017**

Lecture 5: February 3

Lecturer: Thodoris Lykouris — *Scribe: Fikri Pitsuwan*

Price of Anarchy guarantees are vacuous if (a) Pure Nash Equilibria do not exist or (b) players cannot converge to them. This leads to the main focus of this lecture:

1. Do Pure Nash Equilibrium exist in routing games? (Yes)
2. Do some reasonable dynamics converge to the,? (Yes)
3. Do answers to these questions hold for a more general class of games? (Yes but not in general)

We answer these questions using an atomic model of routing game. We then sketch that a similar argument holds for the non-atomic case.

5.1 Atomic Selfish Routing

In non-atomic routing game, each player controls an infinitesimal part of the flow. We now describe the atomic case, where each player controls only one unit of flow.

- set of edges E, each $e \in E$ has a cost function $c_e(x)$
- player i controls one unit of flow from $s_i \to t_i$
- a path that player i takes is $P_i \subseteq E$.
- a flow f is characterized by a list of P_i's.
- $f(e)$ denotes the number of players using edge e.
- cost to player i of flow f is $cost_i(f) = \sum_{e \in P_i} c_e(f(e))$

Definition 5.1 *A flow f is **Pure Nash Equilibrium** if for all i and f',*

$$cost_i(f) \leq cost_i(f'_i, f_{-i})$$

where f_{-i} denotes to the flow induced by all the players but player i and f_i the flow of player i.

Previously, we defined the Price of Anarchy to be the ratio of the cost of the NE flow to the cost of the optimal flow. In atomic routing games, Nash flow costs may not be the same, so we extend the definition.

Definition 5.2 *The **Price of Anarchy** (PoA) is given by*

$$PoA = \frac{\text{cost of worst NE flow}}{\text{cost of optimal flow}}$$

5.2 Existence of Pure Nash Equilibrium in Routing Games

Definition 5.3 *A* ***potential function***[1] *on the flows of an atomic selfish routing game is*

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f(e)} c_e(i)$$

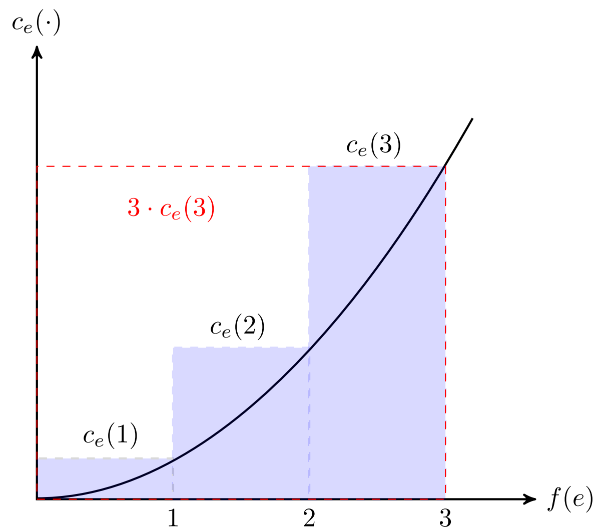

Figure 5.1: e's cost contribution to the potential function (blue) versus cost contribution to total cost (red)

To better understand the potential function, Figure 5.1 considers an edge e with $f(e) = 3$. The cost contribution to the potential function of edge e is $c_e(1) + c_e(2) + c_e(3)$, while the cost that e contributes to the total cost of the flow is $3 \cdot c_e(3)$. Before proceeding, we state the following lemma.

Lemma 5.4 *If player i deviates from path P_i to P_i', resulting in a change from flow f to flow f', we have*

$$\underbrace{\Phi(f') - \Phi(f)}_{\text{change in potential function}} = \underbrace{cost_i(f') - cost_i(f)}_{\text{change in deviator's individual cost}}$$

Proof: Note that for each edge e,

$$f'(e) = \begin{cases} f(e) & \text{if } e \in P_i \cap P_i' \text{ or } e \notin P_i \cup P_i'; \\ f(e) + 1 & \text{if } e \in P_i' \setminus P_i \\ f(e) - 1 & \text{if } e \in P_i \setminus P_i' \end{cases}$$

[1] A general technique for proving existence of Pure Nash Equilbria is via the potential function argument.

We expand the RHS using the definition of potential function,

$$\begin{aligned}
\text{RHS} &= \sum_{e\in E}\sum_{i=1}^{f'(e)} c_e(i) - \sum_{e\in E}\sum_{i=1}^{f(e)} c_e(i) \\
&= \sum_{e\in E}\Big[\sum_{i=1}^{f'(e)} c_e(i) - \sum_{i=1}^{f(e)} c_e(i)\Big] \\
&= \underbrace{\sum_{e\in P_i\cap P_i' \text{ or } e\notin P_i\cup P_i'}\Big[\sum_{i=1}^{f'(e)} c_e(i) - \sum_{i=1}^{f(e)} c_e(i)\Big]}_{=0} + \sum_{e\in P_i'\setminus P_i}\Big[\sum_{i=1}^{f'(e)} c_e(i) - \sum_{i=1}^{f(e)} c_e(i)\Big] + \sum_{e\in P_i\setminus P_i'}\Big[\sum_{i=1}^{f'(e)} c_e(i) - \sum_{i=1}^{f(e)} c_e(i)\Big] \\
&= \sum_{e\in P_i'\setminus P_i}\Big[\sum_{i=1}^{f(e)+1} c_e(i) - \sum_{i=1}^{f(e)} c_e(i)\Big] + \sum_{e\in P_i\setminus P_i'}\Big[\sum_{i=1}^{f(e)-1} c_e(i) - \sum_{i=1}^{f(e)} c_e(i)\Big] \\
&= \sum_{e\in P_i'\setminus P_i} c_e(f(e)+1) - \sum_{e\in P_i\setminus P_i'} c_e(f(e)) \\
&= \sum_{e\in P_i'\setminus P_i} c_e(f'(e)) - \sum_{e\in P_i\setminus P_i'} c_e(f(e)) \quad \underbrace{+\sum_{e\in P_i\cap P_i'} c_e(f'(e)) - \sum_{e\in P_i\cap P_i'} c_e(f(e))}_{\text{add-and-subtract since } f'(e)=f(e) \text{ when } e\in P_i\cap P_i'} \\
&= \sum_{e\in P_i'} c_e(f'(e)) - \sum_{e\in P_i} c_e(f(e)) \\
&= \text{LHS}
\end{aligned}$$

■

Equipped with the Lemma, we are now ready to prove the existence of Pure Nahs Equilibria in atomic selfish routing games.

Theorem 5.5 (Rosenthal (1973)) *Every atomic selfish routing game (with any cost function for edges) has at least one Pure Nash Equilibrium flow.*

Proof:

Let f be a minimizer of Φ, which exists since the number of possible flows in an atomic selfish routing game is finite. By the Lemma, no player can decrease her cost by deviating and thus f is a Pure Nash Equilibrium. ■

Note that we allow f to be a local minimizer of Φ. This implies that Pure Nash Equilibria may not be unique. Existence of Pure Nash Equilibria does not hold for general games, where we can only guarantee existence of a mixed strategy Nash equilibrium (MNE). For example, rock-paper-scissors has Mixed but not Pure Nash Equilibria.

5.2.1 Sketch of Existence of Pure Nash Equilibria in Non-atomic routing games

For non-atomic routing game, we define the potential function as

$$\Phi(f) = \sum_{e\in E}\int_{x=0}^{f(e)} c_e(x)\mathrm{d}x$$

Since cost functions are non-negative, continuous, and non-decreasing for all edges, $\Phi(f)$ is continuous. Also, the space of all flows is compact, so by Weierstrass' extreme value theorem Φ attains a global minimum, which indeed can be shown to correspond to a Pure Nash Equilibrium of the routing game. Note that uniqueness of Pure Nash Equilibria can be established when Φ is strictly convex.

5.3 Best-response dynamics

We have shown that Pure Nash Equilibria exist in routing games. Since Pure Nash Equilbria model how selfish players behave, a natural question is whether there is a reasonable dynamics that leads players to a Pure Nash Equilibrium. The best-response dynamics each player sequentially best respond to all other players gives us the answer.

Definition 5.6 ***Best-response dynamics***

> *While the current flow f is not Pure Nash Equilibrium:*
> *pick an arbitrary player to select a flow that minimizes*
> *her cost given other player's flows, update the outcome to the new flow*

Best response dynamics are guaranteed to converge to Pure Nash Equilibria. The reason is that at every deviation, the cost of the potential function is decreased (by Lemma 5.4). As a result, we can never reach the same outcome twice. Since the number of possible outcomes is finite, this guarantees that best response dynamics will converge after finite steps. Therefore the good news is that *the best-response dynamics converges to Pure Nash Equilibria.*, while the bad news is that this *convergence can be slow.*

5.4 Congestion Games and Potential games

Analysis of existence of Pure Nash Equilibria and best-response dynamics above never use the graphical structure of a routing game. Hence, the same argument holds for a larger class of games called ***congestion games*** defined with the following ingredients.

- a ground set E of congestible elements, each element e has a cost function $c_e(\cdot)$.
- each player i chooses a strategy $P_i \subseteq E$.
- a strategy of the game is then described by $f = (P_1, \ldots, P_i, \ldots)$.
- cost to player i of strategy f is denoted by $cost_i(f)$ and is equal to

$$cost_i(f) = \sum_{e \in P_i} c_e(f_e)$$

We saw above that a routing game has an associated potential function. More generally, we have the following class of games.

Definition 5.7 *Any game where $\Phi(f') - \Phi(f) = cost_i(f') - cost_i(f)$ holds for some Φ is called a* ***potential game****.*

Our analysis shows that every congestion game is a potential game. It turns out that the converse is also true.