CS 6840: Algorithmic Game Theory	**Spring 2017**

Lecture 3: January 30

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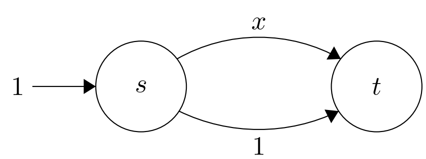
3.1 Price of Anarchy in Routing Games

3.1.1 Review of Last Class

We began with quick review of notation last class (can be found at http://www.cs.cornell.edu/courses/cs6840/2017sp/lecnotes/lec02.pdf)

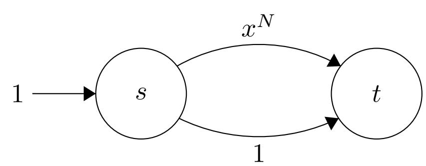
Last class, we used our favorite example.



In this diagram, the **Nash equilibrium** had all travelers take the top edge. The **optimal solution** had half of the travelers on each edge. Thus, the **Price of Anarchy** was 4/3.

3.1.2 A More Difficult Routing Example

We redo the example but make it a little worse.



In this graph, the Nash equilibrium has all travelers take the top route and the cost for each traveler is 1. Thus, if f is the global flow in Nash equilibrium, then the cost ($c(f)$) would be:

$$c(f) = 1$$

For the socially optimal solution, we will let ϵ travelers take the bottom edge and $1 - \epsilon$ take the top edge. If f^* is the socially optimal flow, then the cost would be

$$c(f^*) = (1-\epsilon)(1-\epsilon)^N + \epsilon(1)$$

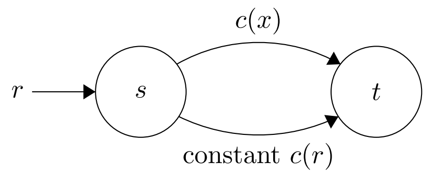
With the above equation, we see that as N goes to infinity, the cost becomes ϵ (because the first term vanishes). Thus, as N approaches infinity,

$$\text{Price of Anarchy} = \frac{c(f)}{c(f^*)}$$

the Price of Anarchy will also approach infinity.

3.1.3 Theorem and Proofs

Now let's redraw the graph to be



Let r be any rate, x be the fraction of travelers on the the top path, and $r-x$ be the fraction on the bottom. The top path has cost $c(x)$, and the bottom has a constant cost $c(r)$.

In a Nash Equilibrium, all travelers will take the top path, so the cost of flow is

$$c(f) = rc(r)$$

In any other flow f^* where $x \neq r$, then cost would be

$$c(f^*) = xc(x) + (r-x)c(r)$$

Let C be a set of monotone and continuous cost functions greater than 0. **Define** $\alpha(C)$ to be

$$\alpha(C) = \sup_{c \in C, 0 \leq x \leq r} \frac{rc(r)}{xc(x) + (r-x)c(r)} \tag{3.1}$$

The numerator in the fraction is the cost of flow in Nash Equilibrium and the denominator is the cost of flow of any other solution (where x fraction of travelers take the top path and $r-x$ take the bottom).

Then the Price of Anarchy on the two-link graph is equal to $\alpha(C)$. This is because the socially optimal solution cost in the denominator would maximize the fraction in $\alpha(C)$ for a given cost function.

Note: The textbook is different in that it does not assume that the cost function is monotone and that it assumes $x, r \geq 0$, instead of $0 \leq x \leq r$. When calculating the Price of Anarchy, both equations yield the same answer. This is because when $x > r$, $\frac{rc(r)}{xc(x)+(r-x)c(r)} < 1$ and is therefore, not the supremum.

To prove this, we will rearrange the ratio:

$$\frac{rc(r)}{xc(x) + (r-x)c(r)} = \frac{rc(r)}{rc(r) + x(c(x) - c(r))}$$

If $x > r$, we see that the RHS is ≤ 1 because c is a monotone function, so the denominator is greater than the numerator.

Theorem 3.1 *If C is a set of monotone and continuous cost functions greater than 0, then the Price of Anarchy on any network with $c \in C$ is less than or equal to $\alpha(C)$, the Price of Anarchy on the two link graph. In other words,*

$$\textit{Price of Anarchy} \leq \alpha(C)$$

To prove this, we have to first prove 2 claims:

Claim 3.2 $\sum_e f^*(e)c_e(f(e)) \geq \sum_e c_e(f(e))f(e)$ *where f is the flow in Nash Equilibrium and f^* is any other flow*

In words, claim 3.2 is stating that given a flow f in Nash Equilibrium, **if we fix the edge costs to those in** f in that graph, the cost of any other flow f^* (using the fixed edge costs) will be greater than or equal to the cost of the Nash Equilibrium flow.

Proof: Imagine cost $c_e(f(e))$ is fixed for all edges e. We can rearrange the LHS and RHS to define the cost in terms of total *path cost* instead of total *edge cost*.

LHS

$$
\begin{aligned}
& \sum_e f^*(e)c_e(f(e)) && \\
=& \sum_p f_p^* c_p(f) && \text{as } c_p(f) = \sum_{e \in P} c_e(f(e)) \\
=& \sum_i \sum_{p \in s_i \to t_i} f_p^* c_p(f) && \text{summing over all paths } p \text{ is equivalent to summing over all paths for each source-sink pair } i \\
\geq& \sum_i \sum_{p \in s_i \to t_i} f_p^* c_i(f) && c_i(f) \leq c_p(f) \text{ b.c. all travelers in Nash Equilibrium are using paths of lowest cost } c_i(f) \\
=& \sum_i (c_i(f) \sum_{p: s_i \to t_i} f_p^*) && c_i(f) \text{ is not path dependent so can be factored out} \\
=& \sum_i c_i(f) r_i && \text{as } r_i = \sum_{p: s_i \to t_i} f_p^* (\text{sum of flow of paths from } s_i \to t_i \text{ is equivalent to } r_i)
\end{aligned}
$$

RHS

$$
\begin{aligned}
& \sum_p f_p c_p(f) \\
=& \sum_i c_i(f) r_i
\end{aligned}
$$

Because the final values in the rearranged LHS and RHS are equivalent and the LHS is greater than this value, then

$$\sum_e f^*(e)c_e(f(e)) \geq \sum_e c_e(f(e))f(e)$$

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Claim 3.3

$$c(f) \leq \alpha(C)c(f^*) \tag{3.2}$$

Claim 3.3 states that the cost of a Nash Equilibrium flow f is less than or equal to the product of $\alpha(C)$ (defined as Equation 3.1) and the cost of flow f^* of any solution.

Proof: Recall that we defined $\alpha(C)$ as

$$\alpha(C) = \sup_{c \in C, 0 \leq x \leq r} \frac{rc(r)}{xc(x) + (r-x)c(r)}$$

Because $\alpha(C)$ is a supremum, then we also know that

$$\alpha(C) \geq \frac{rc(r)}{xc(x) + (r - x)c(r)}$$

For the proof, we will set for each edge e: $r = f(e), x = f^*(e)$. Then the inequality becomes

$$\alpha(C) \geq \frac{f(e)c_e(f(e))}{f^*(e)c_e(f^*(e)) + (f(e) - f^*(e))c_e(f(e))}$$

Multiplying both sides by the denominator and summing over e yields

$$\sum_e f(e)c_e(f(e)) \leq \alpha(C)(\sum_e f^*(e)c_e(f^*(e)) + \sum_e (f(e) - f^*(e))c_e(f(e)))$$

Recall that global cost $c(f) = \sum_e f(e)c_e(f(e))$, so we can replace these values.

$$c(f) \leq \alpha(C)(c(f^*) + \sum_e (f(e) - f*^(e))c_e(f(e)))$$

By Claim 3.2, $\sum_e (f(e) - f*^(e))c_e(f(e))) \leq 0$, so we can remove it from the RHS of the inequality. Therefore,

$$c(f) \leq \alpha(C)c(f^*)$$

■

Proof:

To prove Theorem 3.1, we will use Claim 3.3 which states $c(f) \leq \alpha(C)c(f^*)$.

Dividing both sides by $c(f^*)$ yields the equation

$$\frac{c(f)}{c(f^*)} \leq \alpha(C)$$

f^* was defined to be any flow, which includes the socially optimal flow. If f^* were the socially optimal flow, then $\frac{c(f)}{c(f^*)}$ would be the Price of Anarchy. Therefore,

$$\text{Price of Anarchy} \leq \alpha(C)$$

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