**CS 6840: Algorithmic Game Theory** **Spring 2017**

# Lecture 6: February 6

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## 6.1 Overview

**Last time**, Thodoris talked about the existence of Nash equilibria in the case of *atomic* routing games.

This raises two sets of issues.

The first, which we'll talk about **today**, is that we previously discussed a bound on the price of anarchy in non-atomic games, so now that we've started to talk about atomic games, we'd like to have a similar understanding of the price of anarchy in atomic routing games. So today we're going to talk about the price of anarchy for atomic routing games (more generally, for *congestion games*). Eventually we'll talk about a general technique that yields a lot of the price of anarchy bounds we'll see.

The second, which we'll talk about **next time**, is the problem of existence. Thodoris did not actually prove that some process will *speedily* find the equilibria. We proved that some equilibrium exists, and that some process does find it (by showing it was a local minimum of the potential function). We also want to think about to what extent the price of anarchy is applicable: do we have to first find a Nash equilibrium?

So more generally, we'll talk about the problem of discussing the price of anarchy without first finding a Nash equilibrium. Hopefully these techniques will apply not only when we can find the Nash equilibrium, but also when we cannot.

We'll also spend a little bit of time discussing the complexity of finding equilibria. We're not going to spend a lot of time proving hardness results, though.

## 6.2 Introduction

Consider an atomic routing game with a class $\mathcal{C}$ of cost-functions. Specifically, let's consider linear cost functions of the form $c_e(x) = a_e x + b_e$. What is the price of anarchy?

An atomic routing game consists of the following:

- a directed graph $G = (V, E)$,
- players $i$ with pairs of vertices $s_i$ and $t_i$, and
- strategies, each of which is a path from $s_i \to t_i$.

As before, we define the flow on an edge $f(e)$ as $|\{i : e \in P_i\}|$, that is, the number of players using an edge. And the cost of a particular path $c_p(f) = \sum_{e \in P} c_e(f(e))$.

It turns out there's no beautiful theorem telling us that only two links matter (as with non-atomic routing games). So we're going to get a price of anarchy bound, but it won't be as nice.

As an aside, we can talk more generally about congestion games. In these we are no longer dealing with a graph, specifically, but instead we have some sort of element which is congestible (in the case of routing games, these are the edges). We also have some rule about the things which we are allowed to pick (here, these are the paths, but in a congestion game we might have a tree or a forest).

## 6.3 Finding a bound

Suppose we have some Nash equilibrium using paths $P_1, \ldots, P_k$, with flow $f$, and some optimum of minimum total cost using paths $P_1^*, \ldots, P_k^*$ with flow $f^*$.

Similar to last time, we somehow want to compare the cost of the optimum to the cost of the Nash equilibrium. Consider the first player (going from $s_1$ to $t_1$). They have two options: they can choose either $P_1$ or $P_1^*$. Then we know at least that the first player prefers path $P_1$ to path $P_1^*$ (we know this by the fact that $P_1$ is in a Nash equilibrium).

In fact, if we're thinking about a general congestion game, we don't even know any other information, only that every player $i$ prefers $P_i$ to $P_i^*$. In other words, the flow $\hat{f}$ where $P_i$ has been replaced by $P_i^*$ is actually more expensive for player $i$.

This gives us the following:

$$\sum_{e \in P_i} c_e(f(e)) \leq \sum_{e \in P_i^*} c_e(\hat{f}(e)) \leq \sum_{e \in P_i^*} c_e(f(e)+1) \tag{6.1}$$

How do we get the second (rightmost) inequality? It comes from the fact that $\hat{f}(e) \leq f(e)+1$, that is, the amount of flow on some edge $e$ in the $\hat{f}$ can have increased by at most 1 from the Nash equilibrium, due to player $i$ switching over. And as our cost functions are monotone increasing, this means $c(\hat{f}(e)) \leq c(f(e)+1)$.[1][2]

As we've stated before, we actually don't have any information other than this inequality. So we'd better make good use of it!

Now we sum both sides of the inequality over all players.

What does the left-hand side sum to? We get the following:

$$\begin{aligned} &\sum_i \sum_{e \in P_i} c_e(f(e)) \\ =&\sum_e c_e(f(e)) \cdot |\{i : e \in P_i\}| \\ =&\sum_e c_e(f(e)) f(e) \end{aligned} \tag{6.2}$$

We get the right-hand side by changing the order of the summation (which should look familiar to the previous lecture).

[1]Note that it's possible that the amount of flow on an edge does not increase by 1 in the case that $e \in P_i \cap P_i^*$. But to keep the formulae simple, we will add the plus 1 even it's not needed. This makes the right-hand side an upper bound on the cost of the optimum.

[2]Also note that when the flow was non-atomic, we didn't need the plus 1 (because the addition was infinitesimally small). Aside from that, this is exactly the inequality we had.

How about the right-hand side? Again summing over all players:

$$\begin{aligned}
&\sum_i \sum_{e \in P_i^*} c_e(f(e)+1) \\
=&\sum_e c_e(f(e)+1) \cdot |\{i : e \in P_i^*\}| \\
=&\sum_e c_e(f(e)+1) f^*(e)
\end{aligned} \tag{6.3}$$

... by the same reasoning.

Applying this to (6.1), we get that:

$$\sum_e c_e(f(e))f(e) \leq \sum_e c_e(f(e)+1)f^*(e) \tag{6.4}$$

Now what? Last time, we started looking at the "very funny animal" on the right-hand side, where we are mixing $f(e)$ and $f^*(e)$. We didn't worry about the summation of all of the edges—instead, we just thought about one of the terms and tried to relate it without worrying about the mixing. The analogous strategy here is going to be the following lemma.

## 6.4 Lemma

With some cost function $c(x) \in \mathcal{C}$, we want to be able to say (thinking of $x = f(e), y = f^*(e)$ to get the summands in (6.3)):

$$c(x+1) \cdot y \leq ? \cdot c(y) \cdot y + ? \cdot c(x) \cdot x \tag{6.5}$$

This is what we did last time. If we had something like this, we could expand out our inequality.

It actually turns out that we have the following lemma:

**Lemma 6.1** *For $c(x) = ax + b$, $a, b, x, y \geq 0$:*

$$c(x+1)y \leq \frac{5}{3}c(y)y + \frac{1}{3}c(x)x \tag{6.6}$$

We'll come back to the proof later, and assume it's true for now.

## 6.5 Back to the inequality

Now applying Lemma 6.2 to (6.3) with $x = f(e)$ and $y = f^*(e)$, we obtain the following:

$$\begin{aligned}
&\sum_e c_e(f(e))f(e) \\
\leq &\sum_e c_e(f(e)+1)f^*(e) \\
\leq &\frac{5}{3}\sum_e c(e)(f^*(e))f^*(e) + \frac{1}{3}\sum_e c_e(f(e))f(e)
\end{aligned} \tag{6.7}$$

Subtracting the $\frac{1}{3}$ term from both sides of the inequality (as it's just a multiple of the left-hand side), we get that:

$$\frac{2}{3}\sum_e c_e(f(e))f(e) \leq \frac{5}{3}\sum_e c_e(f^*(e))f^*(e) \tag{6.8}$$

... where the left-hand side is the cost of the Nash equilibrium, and the right-hand side is the cost of the optimum.[3]

So we have that $c(f) \leq \frac{5}{2}c(f^*)$. Note that the price of anarchy in the non-atomic case was $\frac{2}{3}$, so this is a significant increase!

## 6.6 Proof of the lemma

Let's return to the lemma. Here is the lemma again:

**Lemma 6.2** *For $c(x) = ax + b$, $a, b, x, y \geq 0$ (integers):*

$$c(x+1)y \leq \frac{5}{3}c(y)y + \frac{1}{3}c(x)x \tag{6.9}$$

Expanding out $c(x)$ in the equality, we want to show that:

$$(a(x+1)+b)y \overset{?}{\leq} \frac{5}{3}(ay+b)y + \frac{1}{3}(ax+b)x \tag{6.10}$$

We'll proceed by considering at the $a$ and $b$ terms in the inequality separately.

First, consider the $b$ terms. Then the inequality says that $by \overset{?}{\leq} \frac{5}{3}by + \frac{1}{3}bx$. This clearly holds because $b$, $x$, and $y$ are all non-negative (for us, $x$ and $y$ are flow amounts).

Now consider the $a$ terms. The inequality says that $a(x+1)y \overset{?}{\leq} \frac{5}{3}ay^2 + \frac{1}{3}ax^2$. We can divide out the $a$'s to get $(x+1)y \overset{?}{\leq} \frac{5}{3}y^2 + \frac{1}{3}x^2$. Multiplying by 3 to simplify some fractions, we end up with $3xy + 3y \overset{?}{\leq} 5y^2 + x^2$.

We can simplify this by completing the square and obtaining $(2y-x)^2 + (y+x)y \overset{?}{\geq} 3y$. As the first term is a square, it's sufficient that $x + y \geq 3$ for the inequality to hold. We can check the remaining cases by hand, as $x$ and $y$ are both integers.

## 6.7 General strategy

Looking back, what did we do? We wrote down the inequality, which was the only information we had, and worked from there, similar to what we did last time.

[3] We like that we end up with $\frac{1}{3}$ on the right-hand side because it lets us subtract it and get a positive multiplier on the LHS. (A negative multiplier wouldn't be good.)

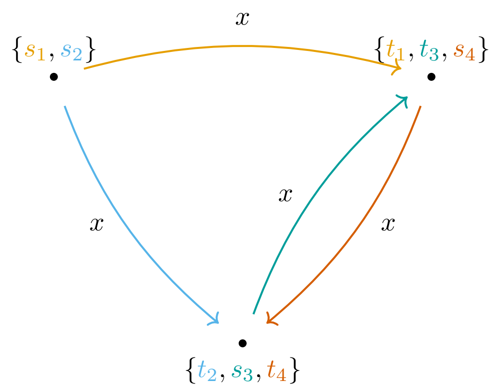


Figure 6.1: Optimum flow of cost 4.

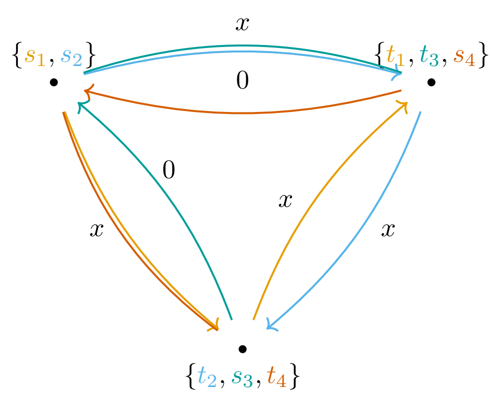


Figure 6.2: A bad flow (and a Nash equilibrium) of cost 10.

## 6.8 A worst-case example

It turns out that the upper bound on the price of anarchy we obtained can actually be achieved. Consider the graph in Figure 6.8. The source-sink pairs and edges used in the optimum flow have been colored correspondingly. Each player uses one edge.

In Figure 6.8, we show a Nash equilibrium which has cost 10—exactly $\frac{5}{2}$ the cost of the optimum flow. Each player uses two edges, and intuitively they have become "tangled," and cannot become untangled without cooperation.