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| **Algorithmic Game Theory** | **Problem Set 2** |
| **CS 6840 Spring 2014** | **Due Wednesday, March 5th** |

The questions on this problem set are of varying difficulty. For full credit, you need to solve at least 4 of the 5 problems below. A full solution for each problem includes proving that your answer is correct. If you cannot solve a problem, write down how far you got, and why are you stuck.

You may discuss the questions with other students, but need to write down the solution yourself. Please acknowledge the students you discussed the questions with on your write-up. You may use any fact we proved in class without proving the proof or reference, and may read the relevant chapters of the book. However, you may **not use other published papers, or the Web to find your answer**.

Solutions can be submitted on CMS in pdf format (only). Please type your solution or write extremely neatly to make it easy to read. If your solution is complex, say more than about half a page, please include a 3-line summary to help us understand the argument.

We will post answers to questions on Piazza.

**(1)** Consider the non-atomic congestion game problem we have been discussing in class (defined on Monday, February 12th). For notational simplicity, here we will assume that all players have different source-sink pairs, so any path $P$ is a strategy for at most one player. We defined Nash equilibria as a solution $f$ with congestion $x$ on the edges such that for all player types $i$ and all strategies $P, Q \in \mathcal{S}_i$, if $f_P > 0$ then $\sum_{e \in P} c_e(x_e) \leq \sum_{e \in Q} c_e(x_e)$.

A maybe more intuitive definition would be as follows. A solution $f$ is a Nash equilibrium, if the following holds. For all player types $i$ and all strategies $P, Q \in \mathcal{S}_i$, if $f_P > 0$ and any $0 < \delta \leq f_P$ if we define a other solution $\hat{f}$ by setting

$$\hat{f}_R = \begin{cases} f_P - \delta & \text{if } R = P \\ f_Q + \delta & \text{if } R = Q \\ f_R & \text{otherwise} \end{cases}$$

Let $x$ and $\hat{x}$ denote the congestion of the solutions $f$ and $\hat{f}$ respectively. The alternate definition states that the flow $f$ is at equilibrium if $\sum_{e \in P} c_e(x_e) \leq \sum_{e \in Q} c_e(\hat{x}_e)$ for all choices of $P$, $Q$ and $\delta$ as above.

This definition considers the solution $f$, where a very small $\delta > 0$ amount of players using strategy $P$, and wonders if this small amount of players would be happier of they switch to another strategy $Q$. Here we model players being non-atomic by allowing arbitrary small amounts of players to switch, but we do not allow "zero" amount to switch, as it is less clear what that means.

Show that the two definitions are the same if the cost functions $c_e(x)$ are monotone increasing (nondecreasing) and continuous. Is this also true for functions $c(x)$ that are not necessarily continuous? how about functions that are not monotone nondecreasing (i.e., that can decrease).

**(2)** We considered in class the notion of $(\lambda, \mu)$-smooth delay functions. For a nonatomic congestion game, a class of functions is $(\lambda, \mu)$-smooth if for all costs $c(x)$ in this class, and any two congestions $x^*$ and $x$ we have that

$$x^* c(x) \leq \lambda x^* c(x^*) + \mu x c(x).$$

(a) Show that the class of all nonnegative monotone increasing (nondecreasing) functions is $(1,1)$-smooth.

(b) We have seen in class (on Friday, February 14) that $(\lambda,\mu)$-smooth for a $\mu < 1$ implies that the corresponding non-atomic congestion game has price of anarchy at most $\lambda/(1-\mu)$. Unfortunately the general bound in part (a) has $\mu = 1$. Show that the above class cannot be $(\lambda,\mu)$-smooth for any constants $\lambda$ and $\mu < 1$ by showing that the price of anarchy can be arbitrarily high. (Hint: enough to consider routing games on a network with two parallel edges, one with $c_e(x) = 1$. By setting the delay of the other edge appropriately, you can achieve arbitrarily high price of anarchy.)

(c) Show that the following holds using (a). For any equilibrium solution $f$ that with rate $r_i$ for user type $i$ and any solution $g$ that satisfies rates $(1+\delta)r_i$ for each player type $i$, we can bound the total delay of $f$ in terms of the delay in $g$. More formally, if $x$ denotes the congestion of flow $f$ and $y$ denotes the congestion of $g$, prove a bound of the form

$$\sum_e x_e c_e(x_e) \leq F(\delta) \sum_e y_e c_e(y_e)$$

for some function $F(.)$ that is defined for all $\delta > 0$ (but can approach infinity as $\delta$ goes to 0).

(d) The following class of cost functions are often used to model capacities. Assume each edge $e$ has two parameters $a_e$ and $u_e$, and let $c_e(x) = \frac{a_e}{u_e - x}$. Note that this function has $c_e(0) = \frac{a_e}{u_e}$, and it models an edge with capacity $u_e$, as cost goes to infinity as the congestion approaches $u_e$.

With this cost function, the methods used so far can help understand the tradeoff between two options in improving networks congestions get too high: (i) increase the capacity of the edges or (ii) improve the routing of the flow. To do this consider your bound from (c) for this class of functions. Compare the cost of an equilibrium flow $f$ to a flow $\hat{g}$ routing the demands $r_i$ in a network with a scaled down capacity $\hat{u}_e = \frac{u_e}{1+\delta}$ for each edge $e$. Give a bound comparing the cost of $f$ and $\hat{g}$.

**(3)** Consider a finite cost minimization game. Assume the game has $k$ players, and the outcomes for players on any choice of strategy vectors $s$ is a cost $c_j(s)$ for player $j$, where the player's goal is to minimize this cost. We showed that if the game is $(\lambda,\mu)$-smooth, meaning for any strategy vector $s$, and a strategy vector $s^*$, that minimizes the total cost $\sum_j c_j(s^*)$ the following inequality holds:

$$\sum_j c_j(s_j^*, s_{-j}) \leq \lambda \sum_j c_j(s^*) + \mu \sum_j c_j(s).$$

We have seen that if a game is $(\lambda,\mu)$-smooth for some $\mu < 1$, then the price of anarchy is bounded by $\lambda/(1-\mu)$, and this bound also applies to the quality of all coarse correlated equilibria. Show that a slightly weaker bound also applies for approximate equilibria. Concretely, let $p$ be a probability distribution of play, a course correlated equilibrium requires that $E_p(c_i(s)) \leq E_p(c_i(s_i^*, s_{-i}))$ for all players $i$ and all strategies $s_i \in \mathcal{S}_i$. An approximate equilibrium requires instead that $E_p(c_i(s)) \leq (1+\epsilon)E_p(c_i(s_i^*, s_{-i}))$ for a small error parameter $\epsilon > 0$. For a given $\lambda$ and $\mu$, how small does $\epsilon$ have to be to make sure your bound applies?

**(4)** Hotelling games is a general class of games when $k$ providers compete for a set of customers. Here we use the following simple case: $G$ is a graph on $n$ vertices. There are $k$ providers, and each provider selects one of the nodes of the game, you can think of the location as a souvenir stand. Each node $v$ in the graph has $n_v > 0$ customers (tourists). Once the sellers selected their locations, each costumer selects the closest seller. In case of ties divide the $n_v$ costumers uniformly among the closest sellers (OK if the fractions are not integers, as we can think of this as the expected number of customers). The goal of the sellers is to attract as many buyers as possible. Let $N_i$ be the total number of customers who selected seller $i$. In this game the traditional social welfare is not a good measure, as we assumed all customers choose a seller, and hence $\sum_i N_i = \sum_v n_v = N$. Instead we will look at a fairness measure, $\min_i N_i$. Clearly this minimum cannot be any higher than $N/k$ in any outcome.

(a) In this game the utility of a player $i$ is between $[0, N]$. The weighted majority algorithm assumed utilities are in the range $[0, 1]$. Show how to adopt the weighted majority algorithm for this game.

(b) Show that that at a pure Nash equilibium (if there is any) all players are guaranteed to get utility (number of customers) at least $N/2k$.

(c) Show that if we play this game repeatedly, and a player $i$ player used a no-regret algorithm, than this player is guaranteed to get average utility (number of customers) at least $N/2k$, independent of the strategies used by other players. More precisely, assume that the player has small total regret over $T$ steps at most $\epsilon TN$, then he/she is guaranteed an average value at least $N/2k - f(\epsilon)$, where $f(\epsilon)$ goes to zero as $\epsilon$ goes to zero.

**(5)** Consider $n$ identical machines and $m$ selfish jobs (the players). Each job $j$ has a processing time $p_j$. Once jobs have chosen machines, the jobs on each machine are processed serially from shortest to longest. (You can assume that the $p_j$s are distinct.) For example, if jobs with processing times 1, 3, and 5 are scheduled on a common machine, then they will complete at times 1, 4, and 9, respectively. The following questions concern the game in which each player $j$ chooses a machine in order to minimize its completion time $C_j$, and the objective function of minimizing the sum $\sum_j C_j$ of the jobs completion times.

(a) Define the rank $R_j$ of job $j$ in a schedule as the number of jobs on $j$th machine with processing time at least $p_j$ (including $j$ itself). For example, if jobs with processing times 1, 3, and 5 are scheduled on a common machine, then they have ranks 3, 2, and 1, respectively. Prove that in these scheduling games, the objective function value of an outcome can also be written $\sum_j R_j p_j$.

(b) Prove that the following algorithm produces an optimal outcome: (i) sort the jobs from largest to smallest; (ii) for $i = 1, 2, \dots, m$, assign the $i$th job in this ordering to machine $i$ mod $n$ (where machine 0 means machine $n$).

(c) Prove that for every such scheduling game, the expected objective function value of every coarse correlated equilibrium is at most twice that of an optimal outcome.

[Hint: Prove that these scheduling games are $(2, 0)$ smooth with the definition of smoothness from the lecture on Wednesday, February 19th.]