CS6840 - Algorithmic Game Theory (3 pages)	Spring 2014
March 19 - Smoothness in Multiple Items Auction Games	
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1 Review:

Definition. An auction is (λ, μ)- smooth if $\exists s^*$, s.t. for all s:

$$\Sigma_i u_i(s_i^*, s_{-i}) \geq \lambda OPT - \mu \Sigma_i p_i(s).$$

Smooth auctions: Set up:

- $o(s)$: outcome
- $v_i(o)$: value of player i. $OPT = \max_o \Sigma v_i(o)$
- $u_i(s) = v_i(o(s)) - p_i(s)$
- $p_i(s) =$ ith payment

Last Time: Smoothness for single item 1st price auction.

Theorem 1. All pay single item auction is $(\frac{1}{2}, 1)$-smooth for any distribution of values.

Proof. : Let $i^* = \arg\max_i V_i$. Let $s_j^* = 0$ for $j \neq i^*$ and s_i^*: randomly chosen according to uniform distribution in $[0, v_i]$. For $j \neq i^*$:

$$u_j(s_j^*, s_{-j}) \geq 0;$$

for $j = i^*$, let $p = \max_{j \neq i^*} s_j$, then:

$$\begin{aligned} u_{i^*}(s_{i^*}^*, s_{-i^*}) &\geq -E(s_i^*) + v_{i^*} Pr(i^* \text{ wins}) \\ &= -\frac{v_{i^*}}{2} + v_{i^*}(\frac{v_{i^*} - p}{v_{i^*}}) \\ &= 0.5 v_{i^*}^* - p \\ &\geq 0.5 v_{i^*}^* - \Sigma_j p_j(s) \end{aligned}$$

Sum up over all i, we get:

$$\Sigma_i u_i(s_i^*, s_{-i}) \geq \frac{1}{2} OPT - \Sigma_i p_i(s)$$

2 Multiple Items:

2.1 Set up for today:

- Unit demand bidders

- Items on sale: Ω
- Players: $1, ..., n$
- Player i has value $v_{ij} \geq 0$ for item j
- $A \subset \Omega$, player i's value for set $A \neq \emptyset$ is $\max_{j \in A} v_{ij}$ (there is free disposal).

2.2 Smoothness

Today: each item is sold on first price.

VCG Mechanism: uses OPT assignment. First price auction uses opt assignment in analysis, but not on mechanism.
Max value matching (optimal matching): $\max_M \Sigma_{(i,j)\in M} v_{ij}$, M represents a Matching.

Theorem 2. 1st price multiple items auction is $(\frac{1}{2}, 1)$-smooth (also $(1 - \frac{1}{e}, 1)$-smooth).

Proof. Take optimal matching M^*. If $(i, j) \in M$ (player i,item j), then bid $s^*_{i^*} = \frac{v_{ij}}{2}$ for item j and bid 0 for all other items. If i is unmatched in M, bid 0 on all items.

If i unmatched,

$$u_i(s^*_i, s_{-i}) \geq 0;$$

Else, $(i, j) \in M$,

$$u_i(s^*_i, s_{-i}) \geq \frac{v_{ij}}{2} - p_j(s).$$

$p_j(s)$ is price for item j on bids s. (This is because if player i wins item j, $u_i(s^*_i, s_{-i}) = \frac{v_{ij}}{2}$; if player i loses item j, item j's price $p_j(s)$ is $\geq \frac{v_{ij}}{2}$.) Sum over i:

$$\Sigma_i u_i(s^*_i, s_{-i}) \geq \frac{1}{2}\Sigma_{(i,j)\in M} v_{ij} - \Sigma_{j\in A} p_j(s) = \frac{1}{2}OPT - \Sigma_j p_j(s)$$

($p_j = 0$ if item j not in assigned).

Corollary 3. Nash equilibrium s for full information game satisfies:

$$SW(s) \geq \frac{\lambda}{\max\{1, \mu\}} OPT.$$

Want Bayesian version:

Option 1: s^*_i depends only on v_i (ith valuation). We used it in single item 1st price auction. Doesn't apply to either "all-pay" of auctions with multiple items.

3 Next Time:

Theorem: smooth game $\rightarrow$ Bayesian PoA small

2nd price auction