CS 6840 Algorithmic Game Theory March 7, 2012

Lecture 20: Revenue Equivalence

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Last time, we looked at the following auction game:

- Each player i has a private value v_i independently drawn from distribution $\mathcal{F}_i$ (which is known publicly).
- The game asks each player i for a bid $b_i(v_i)$.
- The auctioneer determines the *allocation* X_i and *price* P_i for each player i.

We consider X_i and P_i as random variables since they are determined by player values v_i. For each player i, define $F_i(v) := \mathbb{P}[v_i \le v]$. Then the distribution $\mathcal{F}_i$ can be sampled as follows.

1: Sample q uniformly from $[0, 1]$.
2: Compute $v_i(q)$ such that $\mathbb{P}[v_i > v_i(q)] = q$, namely $v_i(q) = F_i^{-1}(1 - q)$.

Now we look at an auction game with only one player whose value v is drawn from some distribution $\mathcal{F}$. If the price of the item is offered as $p = v(q)$, then the auctioneer's revenue equals

$$R(q) = \mathbb{P}[v > v(q)] \cdot v(q) = q \cdot v(q).$$

It holds that $R(0) = R(1) = 0$, and the distribution $\mathcal{F}$ is called *regular* if R is a concave function.

We make the following assumption for today's class:

- F_i functions are continuous, differentiable, and invertible.
- Player's value $v_i \in [0, v_{\max}]$ for all i.
- $\mathcal{F}_i$ distributions are regular.

Recall the following definitions introduced in the last lecture:

$$x_i(v) := \mathbb{E}[X_i \mid v_i = v],$$
$$p_i(v) := \mathbb{E}[P_i \mid v_i = v].$$

We define two new terms using change of variables:

$$\xi_i(q) := x_i(v_i(q)), \qquad \pi_i(q) := p_i(v_i(q)),$$

and the net value for player i is $v_i x_i(v_i) - p_i(v_i)$.

Theorem 1. *If the strategy profile is a Nash equilibrium, then*

(1) $\xi_i(q)$ *is monotone non-increasing in* q*;*

(2)

$$\pi_i(q) = \pi_i(1) - \int_q^1 v_i(r)\xi_i'(r)\,\mathrm{d}r.$$

Proof. (1) follows the fact that $x_i(v)$ is monotone non-decreasing in v. Next we prove (2).

Player i with value $v_i(r)$ can try to bluff to have value $v_i(q)$. In this case, her net value is $v_i(r)\xi_i(q) - \pi_i(q)$. For any q maximizing this value, it holds that

$$[v_i(r)\xi_i(q) - \pi_i(q)]' = v_i(r)\xi_i'(q) - \pi_i'(q) = 0.$$

Since the player's strategy is a Nash equilibrium, picking $q = r$ should maximize the net value. Thus $v_i(r)\xi_i'(r) - \pi_i'(r) = 0$, namely

$$\pi_i'(r) = v_i(r)\xi_i'(r).$$

It follows that

$$\pi_i(1) - \int_q^1 v_i(r)\xi_i'(r)\,\mathrm{d}r = \pi_i(1) - \int_q^1 \pi_i'(r)\,\mathrm{d}r = \pi_i(1) - [\pi_i(r)]_{r=q}^1 = \pi_i(q). \qquad \blacksquare$$

Next we consider how to maximize auctioneer's profit. We assume $\pi_i(1) = p_i(0) = 0$ for all i in the latter part of this lecture.

The expected value for player i is

$$\mathbb{E}[v_i(q)\xi_i(q)] = \int_0^1 v_i(q)\xi_i(q)\,\mathrm{d}q.$$

And expected price paid by this player is

$$\mathbb{E}[\pi_i(q)] = \int_0^1 \pi_i(q)\,\mathrm{d}q = \int_0^1 -\int_q^1 v_i(r)\xi_i'(r)\,\mathrm{d}r\,\mathrm{d}q = -\int_0^1 \int_0^r v_i(r)\xi_i'(r)\,\mathrm{d}q\,\mathrm{d}r = -\int_0^1 r \cdot v_i(r)\xi_i'(r)\,\mathrm{d}r.$$

Let $R_i(r) = r \cdot v_i(r)$. It holds that $[R_i(r)\xi_i(r)]' = R_i'(r)\xi_i(r) + R_i(r)\xi_i'(r)$. Since $R_i(0) = R_i(1) = 0$, we have

$$\int_0^1 [R_i(r)\xi_i(r)]'\,\mathrm{d}r = R_i(1)\xi_i(1) - R_i(0)\xi_i(0) = 0.$$

It follows that

$$\mathbb{E}[\pi_i(q)] = -\int_0^1 r \cdot v_i(r)\xi_i'(r)\,\mathrm{d}r = \int_0^1 R_i'(q)\xi_i(q)\,\mathrm{d}q.$$

And we define $\Phi_i(q) := R_i'(q)$ to be player i's *virtual value*.

Theorem 2 (Myerson '81)**.** *The revenue at Nash equilibrium with allocation function $\xi(q)$ equals expected virtual value*

$$\int_0^1 \Phi(q)\xi(q)\,\mathrm{d}q.$$

Therefore, to maximize auctioneer's revenue (rather than player's total value), we can use the same technique but replacing player's values by their virtual values.

One extra note. If R is concave, then $\Phi = R'$ is monotone decreasing in q and monotone increasing in value. Consider a single-item auction game with n players whose values are drawn independently from the same distribution $\mathcal{F}$. It follows that all players have identical $v(q)$ functions and thus the same $R(q)$. So they have the same virtual value as well. To maximize the revenue, we need to award the item to the player, say i, with maximal virtual value $\Phi(q)$. Since Φ is monotone increasing in value, player i also have maximal value $v(q)$. In addition, we need to make sure that $\Phi(q) > 0$ for player i. In terms of designing an auction, we can set a reserve price r such that $\Phi(v(r)) = 0$ and only hand the item to buyers who are willing to pay more than r.