# CS 6840 - Algorithmic Game Theory

Lecture 29: Individual Item Auctions with Set-Based Valuations

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# 1 Review of Last Lecture

In the last lecture, we started looking at simple auctions where *truthfulness* is not assured. We are interested in analyzing these auctions and in particular the quality of their outcome. In particular, the previous lecture was focused on the application of greedy algorithms in combinatorial auctions and analyzing the price of anarchy of Nash equilibrium in these auctions.

# 2 Individual Item Auction with Set-based values

In this lecture, we will instead look at auctioning individual items, despite having values over sets of these items. We have $n$ bidders, set $S$ of items on sale, each bidder $i$ has value $v_i(A)$ for any subset $A \subseteq S$. Like the previous lecture, we assume that these set functions $v_i$ are monotone non-decreasing with $v_i(\phi) = 0$ for all bidders $i$.

- The mechanism we consider is running separate auctions for each item.
- For each item $a \in S$, each player submits a bid $b_i(a)$ (as always representing bidder $i$'s willingness to pay for $a$). The winner and price charged for each of these items is evaluated separately.
- There are 2 well-studied variations of this mechanism, based on the auction: *First Price Auction* or *Second-Price Auction*

Suppose that the valuations were additive based on the elements in the subset *i.e.,* if

$$v_i(A) = \sum_{a \in A} v_i(a)$$

for all subsets $A \subseteq S$: In this case we can run a second-price auction on each item separately and still be assured of truthfulness and maximum social welfare.

In general we would like to use this mechanism even in situations where the above does not hold. In this lecture, we will assume 2 additional conditions:

1. Full information setting
2. The game has reached a Pure Nash equilibrium [1]

[1] The same analysis can be extended to when players employ learning, as covered in the next lecture.

**NOTE:** What a *truthful bid* means in this setting is not well-defined as although the players have value for sets (and eventually win sets) they bid for individual items only and not for the sets they want.

# 3 First-Price Auction

In the first-price auction setting of this mechanism, each item is awarded to the highest bidder for that item with the price charged equal to that (highest) bid. As per our assumption, suppose the bidders have reached a Pure Nash equilibrium, let bidder $i$ win the auction for all items in set $A_i$ ($A_i = \phi$ is they do not win any of the auctions).

Suppose in the social optimum, bidder $i$ was awarded set $O_i$ instead *i.e.,*

$$O_1, O_2, ...O_n = argmax_{S_1,..,S_n:\forall i,j S_i \cap S_j = \phi}\Big(\sum_{i=1}^{n} v_i(S_i)\Big).$$

Given that the bids $b_i(a)$ (for item $a$ of player $i$) form a pure Nash equilibrium, we would like to analyze the price of anarchy w.r.t the social optimum $O_1, ..O_n$.

Consider the case of a bidder $i$ and an item $a$. Suppose the bidder did not win the item but wanted to win it, the bidder simply needs to raise their bid to $\epsilon$ above $p(a) = \max_j b_j(a)$. We will use this fact in proving the price of anarchy bound.

**ASSUMPTION 1:** Suppose $O_i \cap A_i = \phi; \forall i$.

Consider the following alternate bidding strategy: Bidder $i$ tries to get all the items in $O_i$ instead of those in $A_i$. Hence they instead bid:

- Bid $p(a) + \epsilon$ for all $a \in O_i$
- Bid 0 for all $a \in A_i$

The resulting value of this alternate bidding strategy is: $\geq v_i(O_i) - \sum_{a \in O_i}\Big(p(a) + \epsilon\Big)$. Since the game is in a pure Nash equilibrium, we know that this value is no more than the value the bidder currently achieves which is $v_i(A_i) - \sum_{a \in A_i} p(a)$ (since $p(a) = b_i(a)$ for $a \in A_i$). Thus we get the inequality:

$$v_i(O_i) - \sum_{a \in O_i}\Big(p(a) + \epsilon\Big) \leq v_i(A_i) - \sum_{a \in A_i} p(a)$$

Since this holds for all $\epsilon > 0$, implies that it also holds for $\epsilon = 0$:

$$v_i(O_i) - \sum_{a \in O_i} p(a) \leq v_i(A_i) - \sum_{a \in A_i} p(a)$$

Summing these inequalities over all bidders $i$, we get:

$$\sum_i v_i(O_i) - \sum_i \sum_{a \in O_i} p(a) \leq \sum_i v_i(A_i) - \sum_i \sum_{a \in A_i} p(a) \tag{1}$$

However we note that:

$$\sum_i \sum_{a \in O_i} p(a) = \sum_i \sum_{a \in A_i} p(a) = \sum_{a \in S} p(a)$$

which on substituting in Equation 1 gives us

$$\sum_i v_i(O_i) \leq \sum_i v_i(A_i)$$

This can be interpreted as meaning that the social welfare at a pure Nash equilibrium is equal to the optimal, or in other words, the price of anarchy is 1.

However this result is till contingent on the **Assumption 1** : $O_i \cap A_i = \phi$. To remove this, we note that since this is a Pure Nash equilibrium with a first-price auction mechanism, the highest bid for any item $i$ must have come up from atleast 2 bidders *i.e.,* the winner of the auction (the one with the highest bid) would bid only as much as the second-highest bid otherwise it would not be a Nash equilibrium.

Hence, in this scenario, we have $p(a) = \max_{j \neq i} b_j(a)$ as the bid of $i$. With this additional observation, we can utilize the same identical proof as before and obtain the same result.

**Theorem 1** *For this mechanism with full-information setting in a first-price auction setting, the pure Nash equilibrium is socially optimal.*

# 4 Second-Price Auction

While the first-price setting had the advantage of having the pure Nash equilibrium being socially optimal, it has the drawback that this equilibrium may be harder to find (or may not exist) due to the need for the highest bidder's bid to exactly match that of the second highest, and having these ties broken deterministically (in exactly the manner suitable to the optimal).

However these issues do not come up in a second-price auction setting, which is what we explore next. We will again work with the assumption of a full information setting and analyze the quality of a pure Nash equilibrium[2].

[2] These bounds too can be extended to the scenario with learning.

To start with let us again assume that $A_i \cap O_i = \phi$. Hence if bidder $i$ wanted to win any item $a \in O_i$ he would have instead needed to bid more than $\beta(a) = max_j b_j(a)$ and would have ended up paying $\beta(a)$ as the price for winning that item [3].

Since we have the bidding strategies by the bidders forming a pure Nash, we know that resulting value of any alternate bidding strategy is no more than the current one. In particular, we are interested in the alternate strategy where the bidder $i$ is bidding for elements in $O_i$ with a bid $> \beta(a)$ for $a \in O_i$. The resulting inequality we get is:

$$v_i(O_i) - \sum_{a \in O_i} \beta(a) \leq v_i(A_i) - \sum_{a \in A_i} SecondPrice(a) \leq v_i(A_i)$$

i.e,

$$v_i(O_i) \leq v_i(A_i) + \sum_{a \in O_i} \beta(a)$$

since payments are $\geq 0$. Summing this over all $i$, we get

$$\begin{aligned}
\sum_i v_i(O_i) &\leq \sum_i \sum_{a \in O_i} \beta(a) + \sum_i v_i(A_i) \\
&= \sum_i \sum_{a \in A_i} \beta(a) + \sum_i v_i(A_i) \\
&= \sum_i \Big( \sum_{a \in A_i} \beta(a) + v_i(A_i) \Big) \\
&= \sum_i \Big( \sum_{a \in A_i} b_i(a) + v_i(A_i) \Big) \qquad (2)
\end{aligned}$$

We get the second line by just observing that $\sum_i \sum_{a \in A_i} \beta(a) = \sum_i \sum_{a \in O_i} \beta(a) = \sum_{a \in S} \beta(a)$. The last line comes from the observation that $\beta(a) = max_j b_j(a) = b_i(a)$ as the item was allocated to bidder $i$ in the second -price auction (by the definition of $A_i$).

If we were to remove the assumption that $A_i \cap O_i = \phi$, we can still derive the above inequality by observing that the Nash equilibrium condition gives us:

$$v_i(O_i) - \sum_{a \in O_i / A_i} \beta(a) - \sum_{a \in O_i \cap A_i} SecondPrice(a) \leq v_i(A_i) - Payment \leq v_i(A_i)$$

and the fact that:

$$v_i(O_i) - \sum_{a \in O_i} \beta(a) \leq v_i(O_i) - \sum_{a \in O_i / A_i} \beta(a) - \sum_{a \in O_i \cap A_i} SecondPrice(a)$$

[3]Thus note here that $\beta(a)$ is used to refer to the price you would have to pay to win item $a$.

give us back:

$$v_i(O_i) \leq v_i(A_i) + \sum_{a \in O_i} \beta(a)$$

from which point the proof is identical.

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To obtain a meaningful bound from the Inequality 2, we need to assume that the bids are bounded by the values, which leads us to the below theorem:

**Theorem 2** *For this mechanism with full-information setting in a second-price auction setting, with conservative bidders (i.e., $\forall i, \forall A \subseteq S : \sum_{a \in A} b_i(a) \leq v_i(A)$) the pure Nash equilibrium has a price of anarchy of 2.*