Algorithmic Game Theory	Problem Set 4
CS 6840 Spring 2012	Due Wednesday, April 18th

There are 5 questions on this problem set of varying difficulty. For full credit you should solve 4 of the 5 problems. Solving all 5 results in extra credit. A full solution for each problem includes proving that your answer is correct. If you cannot solve a problem, write down how far you got, and why are you stuck.

You may work in pairs and hand in a shared homework with both of your names marked. You may discuss homework questions with other students, but closely collaborate only with your partner. You may use any fact we proved in class without proving the proof or reference, and may read the relevant chapters of the book. However, you may **not use other published papers, or the Web to find your answer**.

Solutions can be submitted on CMS in pdf format (only). If you have a partner, write both names on the solution, but only upload or submit it once. In any case, please type your solution or write neatly to make it easier to read. If your solution is complex, say more than about half a page, please include a 3-line summary to help us understand the argument.

We will maintain a FAQ for the problem set on the course Web page.

(1) Consider a problem of selling k identical items to n bidders. Assume this time that a bidder may be interested in more than one item. Suppose bidder i is interested in n_i copies of the item, and has value v_i for getting this many items. We assume bidders are "single minded" and have no value for fewer than n_i items. The VCG auction requires to find the subset I of maximum total value $\sum_{i \in I} v_i$, subject to the restriction that $\sum_{i \in I} n_i \leq k$. Here we use variants of the greedy method (discussed in on Monday, March 26)

(a) For this part, we consider only the optimization aspect of the problem. Assume that v_i and n_i are known, and consider the greedy mechanism that sorts agents in decreasing order of v_i/n_i and assigns them items "till the supplies last". Assume that each n_i is small compared to k, say $cn_i \leq k$ for a constant $c > 1$, and show that this algorithm is a $1 - 1/c$ approximation algorithm, that is, finds a solution that has value at least a $\frac{c}{c-1}$ of the maximum possible.

(b) Now consider the greedy algorithm in the game theoretic context. For the remainder of this problem, assume that n_i is public knowledge, and only v_i is private. Consider the following mechanism. Ask players for a bid b_i (willingness to pay), and sort by b_i/n_i, and assigns goods to agents till the supplies last. The traditional definition of critical price would assign the minimal the per-unit price at which the bidder is still getting assigned his n_i unit. Is the greedy mechanism with this price truthful (that is, is bidding the true value always a Nash equilibrium)?

(c) A simpler version of "critical price" charges agent minimal the per-unit price that he/she had to bid to keep his spot in the sorted order. More formally, if agent i requires n_i units, and is right before agent j in the sorted order, than the price i gets changed is $n_i \frac{b_j}{n_j}$. Is this mechanism truthful?

(d) Consider a Nash equilibria of the above mechanism, where we assume $b_i \leq v_i$ for all i. (Note that bidding above v_i is dominated). Show a bound on the price of anarchy for this mechanism. That is, bound the total value $\sum v_i$ of all agents who are assigned their required items in Nash compared to the highest possible. You may want to start with the special case when $n_i = 1$ for all i, and see how this generalizes to agents requiring larger bundles.

(e) Does your analysis of part (d) extend to the case when the quantity n_i is also private, and agents need to announce (q_i, b_i) pairs as bids? Assume there is free disposal, so getting more than n_i items is just as useful as getting exactly n_i items, but agents have no value for less than their required n_i items.

(2) A valuation $v(.)$ function is said to be submodular (or satisfy economy of scale) if for any two sets $A \subset B$, and any element a we have $v(A+a)-v(A) \geq v(B+a)-v(B)$. Assume also that $v(\emptyset) = 0$. Show that all such submodular valuation functions are also fractionally subadditive (can be defined as $v(A) = \max_{x \in X} \sum_{a \in A} x(a)$ for some set of vectors x. Hint: consider additive valuations defined by an ordering of the elements $a_1, \ldots, a_n$ with $x(a_i) = v(\{a_1, \ldots, a_i\}) - v(\{a_1, \ldots, a_{i-1}\})$.

(3) We have seen on Wednesday April 4th that if valuations are fractionally subadditive, and bidders are conservative then in the game of independent item auction all mixed Nash equilibria have welfare at least 1/2 of the maximum possible. This question aims to explore of something similar can also be said about the analogous Bayesian game in which types (valuations) of players are random, with known and independent distributions. In a Nash equilibrium in a Bayesian game the player's strategy can only depend on his own type. Here we will will consider only pure equilibria, i.e., assume that the player's strategy is a deterministic function of his/her type. Let A_i^v denote the allocation i gets at Nash and let O_i^v the optimum allocation for player i when the player types are v. Also let $p_{-i}^v(a) = \max_{j \neq i} b_j^v(a)$ the maximum bid of all players except i for item a and type profile v at Nash. Note that $p_{-i}^v(a)$ is a random variable that depends on the valuations (and the resulting bids).

(a) Consider a fixed valuation vector v and a player i. Let w a random valuation vector with coordinates selected at random according to the Bayesian distribution of valuations. Show that for each vector v we have $v_i(O_i^v) \leq E_{w_{-i}}(v_i(A_i^{v_i, w_{-i}})) + E_{w_{-i}}(\sum_{a \in O_i^v} p_{-i}^w(a))$

(b) Sum the terms $E_{w_{-i}}(v_i(A_i^{v_i, w_{-i}}))$ for all values of i (for a fixed v). Show that the expectation of this sum (over v) is the same as $E_v(\sum_i v_i(A_i^v))$.

(c) Consider the second term $E_{w_{-i}}(\sum_{a \in O_i^v} p_{-i}^w(a))$, and again take sum over i for a fixed valuation v, and then take expectation over v. Show that the expectation is bounded by $E_v(\sum_i v_i(A_i^v))$.

(d) Show that any the expected social welfare in a pure Nash equilibrium in a pure Bayesian Nash is at least 1/2 times the optimum. Does this analysis also work if types are not independent?

(4) Consider the position auction environment with $n = m = 2$ from the lecture on Friday, April 6th with click-through rates 1 and 1/2 on the two slots. Consider running the following first-price auction: The advertisers submit bids $b = (b_1, b_2)$. The advertisers are assigned to slots in order of their bids. Advertisers pay their bid when clicked. Use revenue equivalence to solve for symmetric Bayes-Nash equilibrium strategies s when the values of the advertisers are drawn independent and identically from the uniform distribution as $[0, 1]$.

(Question: can this game have a non-symmetric equilibrium?)

(5) Extend the fair-sharing mechanism to a pair of resources, say with capacity C_1 and C_2 and 3 users. Users 1 and 2 wants only resource 1 and 2 respectively, and have a utility function $U_i(x)$ for the resource, while user 3 wants the same amount of both. So if allocated amounts y_1 and y_2 his/her utility is $U_3(\min(y_1, y_2))$. Assume that U_i is continously differentiable, bf strictly monotone increasing and concave. There are two variants of this game.

Users 1 and 2 offer amount w_i of money for the resource they want, user 3 offers amounts w'_1 and w'_2 for the two resources separately. Each resource is then allocated using fair sharing: $x_1 = C_1 \frac{w_1}{w_1+w'_1}$ for user 1, and $y_1 = C_1 \frac{w'_1}{w_1+w'_1}$, and similarly for resource 2.

(a.) This allocation is very easy to implement, but it possibly allocates different amounts to user 3 of the two resources, which is not good for user 3. Show that this will not happen in equilibrium, that is $y_1 = y_2$ in any equilibrium.

(b.) Give the conditions of equilibrium for this game, and argue that equilibrium must exists.

Even though user 3 will get the same amount of the two resources at equilibrium, one may view this game as awkward, as it can allocate different amounts on the two edge to user 2. An alternate version would be the following.

Users 1 and 2 offer amount w_i of money for the resource they want, user 3 offers amounts w_3 for the two resources combined. We then solve the equation system $w'_1 + w'_2 = w_3$ and $C_1 \frac{w'_1}{w_1+w'_1} = C_2 \frac{w'_2}{w_2+w'_2}$. Then use w'_1 and w'_2 in the fair sharing mechanism, as above.

Note that this system always allocates the same amount of both resources to user 3, but its harder to compute the allocation.

(c.) **Assume that $C_1 = C_2 = C$ and $U_1 = U_2$ for this part.** Give the conditions of equilibrium for this new game, and argue that equilibrium must exists. (Note that without some extra assumption the equilibrium does not always exists.)

(d.) This part is attempting to explore which of the two rules would the players prefer. Give an intuitive explanation of your finding. Consider the **further** special case when the utility of all players is $U_i(x) = x$. What are the optimal allocation, and the Nash allocations in the two different games. Which player prefers which of the rules?