Algorithmic Game Theory	**Problem Set 3**
CS 6840 Spring 2012	**Due Wednesday, March 14th**

There are 6 questions on this problem set of varying difficulty. For full credit you should solve 5 of the 6 problems. Solving all 5 results in extra credit. A full solution for each problem includes proving that your answer is correct. If you cannot solve a problem, write down how far you got, and why are you stuck.

You may work in pairs and hand in a shared homework with both of your names marked. You may discuss homework questions with other students, but closely collaborate only with your partner. You may use any fact we proved in class without proving the proof or reference, and may read the relevant chapters of the book. However, you may **not use other published papers, or the Web to find your answer**.

Solutions can be submitted on CMS in pdf format (only). If you have a partner, write both names on the solution, but only upload or submit it once. In any case, please type your solution or write neatly to make it easier to read. If your solution is complex, say more than about half a page, please include a 3-line summary to help us understand the argument.

We will maintain a FAQ for the problem set on the course Web page.

(1) Consider the setting when a single item is on sale, and agents (players) have a private value for this item, where values are down independently from the same distribution $\mathcal{F}$. In class we considered the first and second price mechanisms for selling an item in this environment. Here consider an "all-pay" auction: as usual all agents submit a bid b_i, and we collect payment $p_i = b_i$ from all. The item is awarded to the highest bidder. Such auctions model the situation when pay is collected as "entry-fee". For example, crowdsourcing is a good example: where we can think of the work you put into participation as a payment. Show that in this setting all-pay auctions also have a symmetric Bayes-Nash equilibrium that has the same revenue (total payment to the auctioneer) as first and second price.

(2) Consider the VCG mechanism in a single dimensional environment, where players private value is characterized by a single number v_i. We have seen that at a Nash equilibrium the outcome $x_i(v)$ for a player i as a function of its value v has to be monotone non-decreasing in its value v, and that the payment p_i is independent of the agents value (depends on the other player's values only). In this question we explore how an agents outcome and payment can depend on other players valuations in the VCG mechanism.

(a) The outcome x_i of an agent i depends also on other agent value v_j. Keeping all other values fixed, is x_i necessarily a monotone increasing or decreasing function of v_j? Prove or give a counter example.

(b) Let p_i be the VCG payment of agent i. In the range of values when x_i is fixed, the price p_i does not depend on agent i's value v_i. In this range, and keeping all other values fixed, is p_i necessarily a monotone increasing or decreasing function of v_j? Prove or give a counter example.

(3) Consider the following single-bidder prior-free pricing game. There is an unknown value $v \in [1, h]$, and you don't have any information on what v may be. If you offer a price $p \leq v$ you get

p dollars and otherwise you get nothing. Design a randomized pricing strategy (i.e., a probability distribution over prices) such that your expected revenue is at least v/α for every v, where α is as close to 1 as you can manage. Prove the best lower bound that you can on what values of α are achievable in this game.

(4) Public Project Problem: A mayor wants to decide whether to build a bridge. The bridge has cost C and (private) value $v_i \geq 0$ for each citizen. The mayor wants to build the bridge if and only if $\sum_i v_i > C$.

(a) Suppose the mayor wants to use the VCG mechanism for this problem. Express the price p_i the players have to pay if the bridge is built using the cost C and values v_i.
(b) Show that the total payment $P = \sum_i p_i \leq C$ in this mechanism.
(c) How small can the ratio $\frac{P}{C}$ get?

(5) [Problem 3.1. in Hartline] In computer networks such as the Internet it is often not possible to use monetary payments to ensure the allocation of resources to those who value them the most. Computational payments, e.g., in the form of proofs of work, however, are often possible. One important difference between monetary payments and computational payments is that computational payments can be used to align incentives but do not transfer utility from the agents to the seller. I.e., the seller has no direct value from an agent performing a proof-of-work computation. We define the residual surplus as the social surplus less the payments.

Consider a single item auction in such environment. Assume there are n agents whose values are drawn from a distribution F that satisfy the monotone hazard rate assumption, i.e., $f(v)/(1-F(v))$ is monotone non-decreasing, as also defined on Monday March 5th in class. Describe the mechanism that maximizes residual surplus, that is maximizes $\sum_i (v_i x_i - p_i)$, where $x_i = 1$ for the agent that receives the good, and 0 for all other agents. Your description should first include a description in terms of virtual values and then you should interpret the implication of the monotone hazard rate assumption to give a simple description of the optimal mechanism.

(6) This problem shows that, for Bayesian-optimal mechanism design, "sufficient competition" can obviate the need for a reserve price.

(a) Consider a distribution $\mathcal{F}$ that is regular in the sense discussed in the lecture of Monday, March 5th, and let $\phi(.)$ denote the corresponding virtual valuation function. Prove that the expected virtual value $\phi(v)$ of a valuation v drawn from $\mathcal{F}$ is zero.
(b) Consider selling $k \geq 1$ identical items to bidders with valuations drawn i.i.d. from $\mathcal{F}$. Prove that for every $n \geq k$, the expected revenue of the Vickrey auction (with no reserve) with $n+k$ bidders is at least that of the Bayesian-optimal auction for $\mathcal{F}$ with n bidders.
[Thus, modest additional competition is at least as valuable as knowing the distribution F and employing a corresponding optimal reserve price.] [Hints: To develop intuition explore the case of $k = n = 1$ and $\mathcal{F}$ uniform on $[0, 1]$. In general, use Myersons characterization of the expected revenue of a truthful auction. Condition on the values of the first n bidders and then argue about the expected impact of the final k bidders on the revenue of the no-reserve Vickrey auction, using part (a).]
(c) In the same setup as (b), assume that $n > 2$ and consider the following alternative mechanism: pick one of the n bidders uniformly at random, say with bid r, and run the Vickrey auction with reserve r on the other $n-1$ bidders. Prove that the expected revenue of this auction is at least $(n-1)/2n$ times that of the Bayesian-optimal auction (with n bidders).