CS 6840 – Algorithmic Game Theory (3 pages)	Spring 2012
Lecture 25 Scribe Notes-Generalization on matroids	
Instructor: Eva Tardos	*Chenhao Tan*

In this lecture, we mainly discussed how to generalize our previous results in single item auction on matroids.

1 Matroid

First we will give the definition of a matroid. A matroid is a set system $(E, \mathcal{I})$ where E is the ground set of elements and I is a set of independent subsets of E.

Definition. Independent sets satisfies:

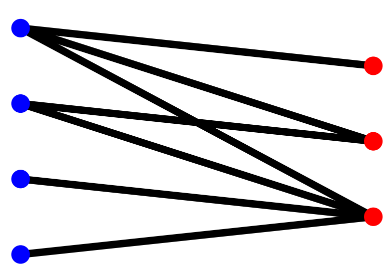
- if A is independent, $B \subset A$, then B is also independent.
- if A and B are both independent, $|A| > |B|$, then $\exists a \in A \backslash B$, so that $B \cup \{a\}$ is independent.

An immediate corollary is that independent sets with maximum size are all with the same size. It is called *rank*.

In the context of auction, independence means that we can sell a service to each element. In an auction, each person s is with value v_s, we sell service to subset A at some price, A must be independent.

Here are some examples.

- Selling single item, the corresponding matroid: 1-element sets and $\emptyset$
- Selling k items, the corresponding matroid: $\leq k$-element sets
- Each player s with a vector $z_s \in \mathbb{R}^d$, the corresponding matroid: linear independence, A s.t. $\{z_s : s \in A |\text{linear independent}\}$
- matchable sets, in a bipartite graph, each client can only buy items connected by an edge, the corresponding matroid: valid matchings.



Now recall the results in single item auction,

(a) sell service in matroid structure, players' values are drawn from identical regular distribution $\mathcal{F}$:

- maximize social welfare
- maximize revenue

(b) (a) with non-regular distribution.

(c) 2-approximation if distribution of v_i is from a regular distribution $\hat{\mathcal{F}}_i$.

(d) ??-approximation if distribution of v_i is from a non-regular distribution $\hat{\mathcal{F}}_i$.

In this lecture, we covered (a) and (b).

2 Generalization on matroids

2.1 Maximizing social welfare

In order to maximize social welfare $\sum_{i \in I} v_i$, select independent set I with Algorithm 1.

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Ask value v_i, sort v_1 ≥ ... ≥ v_n;
I = ∅;
for i = 1, ..., n do
    add v_i to I if I + {i} is independent and v_i ≥ 0;
end
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Algorithm 1: Greedy Algorithm

Theorem 1. Greedy algorithm maximizes $\sum_{v \in I} v_i$ over matroids.

Proof. The proof is from "Approximation in Economic Design" by Jason Hartline.

Let r be the rank of the matroid. Let $I = \{i_1, \ldots, i_r\}$ be the set of elements selected in the surplus maximizing assignment, and let $J = \{j_1, \ldots, j_r\}$ be the set of elements selected by the greedy algorithm. The surplus from serving a subset S of the agents is $\sum_{i \in S} v_i$. Assume for a contradiction that the surplus of set I is strictly more than the surplus of set J, i.e., greedy algorithm is not optimal. Assume the items of I and J are indexed in decreasing order. Therefore, there must exist a first index k such that $v_{i_k} > v_{j_k}$. Let $I_k = \{i_1, \ldots, i_k\}$ and let $J_{k-1} = \{j_1, \ldots, j_{k-1}\}$. Applying the augmentation property to sets I_k and J_{k-1} we see that there must exist some element $i \in I_k \backslash J_{k-1}$ such that $J_{k-1} \cup \{i\}$ is feasible. Of course, $v_i \geq v_{i_k} > v_{j_k}$ which means agent i was considered by greedy algorithm before it selected j_k. According to the definition of independent sets, $J_{k-1} \cup \{i\}$, when i was considered by greedy algorithm and it was feasible. By definition of the algorithm, i should have been added; this is a contradiction. □

This is VCG. Ask players about values, find the solution that maximizes social welfare and use price characterization to set price.

Note in this greedy algorithm, **no assumption** is needed.

2.2 Maximizing expected revenue

The goal is to maximize $E[\sum_{i \in I} \phi_i(v_i)]$, $\phi_i(v_i)$ is the virtual value of player i when value is v_i.

If distributions are identical, $\phi_i(v) = \phi(v)$, the goal becomes maximizing $E(\sum_{i \in I} \phi(v_i))$.

If the distribution is regular, then $\phi(v)$ is monotone non-decreasing. If $v_1 \geq \ldots \geq v_n$, then $\phi(v_1) \geq \ldots \geq \phi(v_n)$.

We can just run the same greedy algorithm 1 with reserve price r such that $\phi(r) = 0$.

2.3 Non-regular distribution

As what we did in single item auction, we need to consider ironed virtual value $\bar{\phi}(q) = \bar{R}'(q)$, where $\bar{R}(q)$ is the smallest concave function no smaller than the revenue with sale prob q, $R(q) = qv(q)$. $\bar{\phi}(q)$ is monotone non-increasing in q, monotone non-decreasing in value v.

As we have shown in last lecture, the goal is to maximize revenue $E(\sum_{i \in I} \bar{\phi}(v_i))$.

Note if $v_1 \geq \ldots \geq v_n$, then $\bar{\phi}(v_1) \geq \ldots \geq \bar{\phi}(v_n)$, set reserve price at $\bar{\phi}(r) = 0$. One important property we needed last time is that in regions where $\bar{R}_i(q) > R_i(q)$ allocation x_i is fixed. (If true, $E(\sum \bar{\phi}_i(v_i)) = E(\sum \phi_i(v_i))$). In this region $\bar{\phi}_i(v)$ is constant.

Then the algorithm needs changing a little:

Sort by $\bar{\phi}$, choose between identical $\bar{\phi}$ values by some consistent rules not using the order of values, for example random or alphabetically.