CS 6840 – Algorithmic Game Theory (3 pages) Spring 2012

Lecture 41 Scribe Notes

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1 Lecture 41 – Friday 27 April 2012 – High Dimension Sperner Lemma

Last time we proved 2 dimensional version of Sperner lemma. Today we will prove Sperner lemma in high dimensional case.

1.1 Introduction

- Let $\Delta = \{\sum_i^n x_i = 1, x_i \geq 0, \text{ for all } i = 1, ..., n\}$ be a simplex in n dimension
- It is subdivided to little simplices of side δ
- Color each vertex colors $\{1, ..., n\}$, such that side with $x_i = 0$ does not use color i.

Lemma 1 (Sperner lemma)**.** There exist odd number of multicolored simplices (using all colors)

We proved this statement when $n = 2, 3$. We will use this to prove high dimensional case.

Theorem 2 (Brouwer's)**.** If $f : \Delta \to \Delta$ is a continuous function, then there exists x such that $f(x) = x$.

A function is continuous if $\forall \varepsilon, \exists \delta$, such that $\forall p, q$, $\|p - q\| \leq \delta \Rightarrow \|f(p) - f(q)\| \leq \varepsilon$. Consider $p = (x_1, ..., x_n)$ and $f(p) = (x'_1, ..., x'_n)$. If $p = f(p)$, we are done and find a fixed point. If $p \neq f(p)$, there exists i so that $x'_i < x_i$, then we color p by i. This satisfies Sperner lemma assumption and hence we have a multicolored triangle.

The goal is to show $\|p - f(p)\|$ is small, then it is (almost) a fixed point. Let us bound $x'_i - x_i$ first. As shown in Figure 1, we use a neighboring point q colored i, $q = (y_1, ..., y_n)$, and $f(q) = (y'_1, ..., y'_n)$, we have

$$\begin{aligned}
x'_i - x_i &\leq x'_i - x_i + (y_i - y'_i) && (q \text{ is colored } i, \text{ so } y'_i < y_i) \\
&= (x'_i - y'_i) + (y_i - x_i) \\
&\leq \|f(q) - f(p)\| + \|p - q\| \\
&\leq \varepsilon + \delta && (\text{By continuity definition})
\end{aligned}$$

Note the $\sum_i^n x_i = 1$, since the increase is bounded, the decrease is also bounded. This implies, if we choose L_1 norm, $\|p - f(p)\|_1 \leq 2n(\varepsilon + \delta)$. Since we are interested in the existence proof, select a sequence of ε_k and δ_k so that $\max(\varepsilon_k, \delta_k) \leq 2^{-k}$. Thus, we could set $\varepsilon = 2^{-k}$ and pick δ from continuity so that $\delta \leq 2^{-k}$, then a point p_k satisfies $\|f(p_k) - p_k\|_1 \leq 2n2^{-k}$.

Consider sequence $p_1, ..., p_k, ...$, and take limiting point, use simplex Δ, which is bounded and closed, this implies there is a subsequence going to limit p.

Claim 1. p is a fixed point.

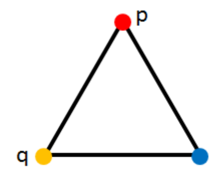

Figure 1: Corner of multicolored simplex of side δ.

Proof. By contradiction, $p \neq f(p)$. Choose the p_k that is close to p, then $f(p_k)$ is also close to p. Thus $f(p)$ can not be too far away from p. Assume $\|p - f(p)\|_1 = 3\varepsilon$ in Figure 2. We can choose a large enough k so that $\|f(p_k) - p_k\|_1 \leq 2n2^{-k} < \epsilon$. Thus $\|p - f(p_k)\|_1 < 2\varepsilon$. Contradiction. □

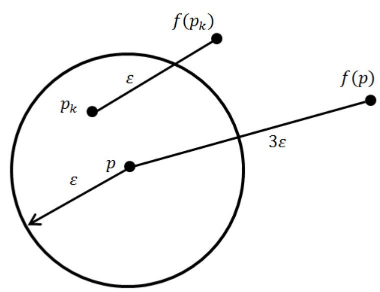

Figure 2: Illustration of p is a fixed point.

1.2 Algorithmic Sperner Lemma

Algorithm for Sperner lemma can be used to find p such that $\|p - f(p)\|_1 \leq \varepsilon$; select small enough δ, subdivide, color and find multicolor simplex. Let us describe the algorithmic Sperner lemma.

- Input: coloring algorithm/circuit. Each input $(x_1, ..., x_n)$ is in binary.
- In our case, we use function f and compute $f(x_1, ..., x_n) = (x'_1, ..., x'_n)$. Color i if $x'_i < x_n$.
- Output: multicolored Δ or violation of promise $(x_1, ..., x_n)$ with $x_i = 0$ does not use color i.

1.3 Proof of Sperner Lemma in $n \geq 2$ Dimension

By induction face $x_n = 0$ has odd number of multicolored $(n-1)$ dimension simplices (color $\{1, ..., n-1\}$). Create a graph of two kinds of vertices:

1. Multicolored simplex in $n - 1$ dimension.
2. All full dimensional little simplices

Add edges if two vertices share on $n - 1$ dimensional simplex colored $1, ..., n - 1$.

Observation. Type 1 vertices have degree 1. Type 2 vertices have degree 1 if they are multicolored, degree 0 if not all $1, ..., n - 1$ show up, degree 2 if some color i repeats.

This implies there are even number of multicolored simplices as $\sum$ degrees is even. Given the induction assumption that there are odd number type 1 node (whose degree is 1), we can conclude that there are odd number of multicolored simplices in n dimension.