CS 6840 – Algorithmic Game Theory (4 pages) Spring 2012

Prize of Anarchy in Generalized Second Price Auction

Instructor: Eva Tardos *Anshumali Shrivastava (as2446)*

1 Lecture 33 – Monday 9 April 2012 - Prize of anarchy in Generalized Second Price (GSP) auction

1.1 Recall from last lecture

Setting:

n players : each with private value v_i (dollars per click)
n slots : available slots for ads with click-through rates $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$

GSP Game:

Strategy for player i: bid b_i
Sort Players by bids.
Let $\pi(j)$ = player wth j-th highest bid (i.e player that will go to slot j)
and $\sigma(i) = \pi^{-1}(i)$ denote the slot player i goes to.
Charge next highest bid

Utilities :

$u_i = \alpha_j(v_i - b_{\pi(j+1)})$ if player i gets slot j. It can also be written as
$u_i = \alpha_{\sigma(i)}(v_i - b_{\pi(\sigma(i)+1)})$

The social welfare is defined as

$$SW = \sum_i u_i + revenue = \sum_i \alpha_i v_{\sigma(i)}$$

If $v_1 \geq v_2 \geq .. \geq v_n$ then

$$Opt = \sum_i \alpha_i v_i$$

Last class we showed

- existence of equilibrium
- Price of Stability = 1 (i.e the equilibrium is in fact socially optimal).

1.2 Price of Anarchy (PoA) Results

This class will focus more on PoA i.e on the wost case guarantee for any equilibrium. We start by stating a theorem in this setting

Theorem 1. Price of Anarchy is ≤ 1.282

In other words for any α, v and for any pure Nash bids b with resulting allocation π we have

$$\sum_i \alpha_i v_{\pi(i)} \geq \frac{1}{1.282} \sum_i \alpha_i v_i$$

The proof can be found in "On the efficiency of equilibria in generalized second price auctions (http://arxiv.org/abs/1201.6429)". It is easy to prove a factor of two, the proof is very much similar to the proof given for the Second Price Item Auctions two lectures ago. It follows the same philosophy, that if you don't get optimal, you try to deviate your bid to get the optimal and then apply equilibrium inequality, sum all of them up, and get the desired result.

Today we will focus on the Bayesian version of this setting and the price of anarchy results. Specifically we will look at the following

- Example of PoA bounds for Bayesian Games
- Application of the smoothness framework

1.3 Bayesian version of GSP

Bayesian Setting : $v_i \sim F_i$: i.e the value for player i comes from the distribution F_i, here we don't assume anything about $F_i s$, they may be correlated.
Strategies are bidding functions $b_i(v_i)$, we will assume $b_i(v_i) \leq v_i$.
So the game goes something like this

- Step 0 : nature chooses v_i
- Step 1: players submit bids $b_i(v_i)$
- Step 2: run GSP on bids $b_i(v_i)$

Bayes Nash Equilibrium in this case is a situation where, even after realizing the outcome of v_i the player i does not want to change. This condition can also be written as

Definition. Bayes Nash Equilibrium Condition

$$\mathbb{E}[u_i(b_i(v_i), b_{-i}(v_{-i}))|v_i] \geq \mathbb{E}[u_i(b'_i(v_i), b_{-i}(v_{-i}))|v_i]$$

i.e the conditional expectation of utility, for a particular player, under the equilibrium bidding function is more (or at least as good as) than the conditional expectation using any other bidding function.
For notation convenience lets define $\tau(i)$ to be the slot the ith player goes to in the optimal allocation.
The expected Social Welfare (SW) in this setting can be written as

$$\mathbb{E}[SW] = \mathbb{E}[\sum_i \alpha_i v_{\pi(i)}]$$

It is worth noting here that both π and v are random Also using the τ we can write the expected value of Optimum to be

$$\mathbb{E}[Opt] = \mathbb{E}(\sum_i \alpha_{\tau(i)} v_i)$$

We are interested in the ratio
$$\frac{OPT}{SW} = \frac{\mathbb{E}[\sum_i \alpha_{\tau(i)} v_i]}{\mathbb{E}[\sum_i \alpha_i v_{\pi(i)}]}$$
.

Theorem 2. Bayes Nash PoA ≤ 2.927

The proof is messy, because it uses a dirty distribution with many parameters and then finally optimizes for the parameters, using linear programming, to get this weird number 2.927.

For simplicity we demonstrate the proof for a factor of 4.
Proof for bound 4 The idea is that we let the players deviate their bid to some artificial function, (easy to handle) and then look at the two cases, where in one case his bid is sufficient to get the optimal and in other it isn't, and then use the Nash inequality to complete the proof.

Consider that instead of using the equilibrium bid $b_i(v_i)$, the player deviates to $b'_i(v_i) = \frac{v_i}{2}$. We will use the usual notation for all other player except i sticking to their old bids as $b_{-i}(v_{-i})$. Let p_i be the price paid by player i under this bid There are two possibilities :

- $\frac{v_i}{2}$ is strong enough to fetch him slot $\tau(i)$ (the optimum) or better. Which means
$$u_i(b'_i(v_i), b_{-i}(v_{-i})) \geq \alpha_{\tau(i)}(v_i - p_i)$$
the worst possible payment in this case is $\frac{v_i}{2}$. He is suppose to pay the next highest bid which in worst case could be equal to his bid. So we can write
$$u_i(b'_i(v_i), b_{-i}(v_{-i})) \geq \alpha_{\tau(i)}(v_i - \frac{v_i}{2}) = \frac{\alpha_{\tau(i)} v_i}{2}$$
- suppose that $\frac{v_i}{2}$ is not enough to get slot $\tau(i)$ or better, this can happen only when the $\tau(i)$th highest bid is greater than $\frac{v_i}{2}$. i.e
$$b_{\pi(\tau(i))} \geq \frac{v_i}{2} \rightarrow v_{\pi(\tau(i))} \geq \frac{v_i}{2}$$
multiplying by $\alpha_{\tau(i)}$ gives
$$\alpha_{\tau(i)} v_{\pi(\tau(i))} \geq \alpha_{\tau(i)} \frac{v_i}{2}$$

In either case we can write the inequality as all the quantities are positive
$$u_i(b'_i(v_i), b_{-i}(v_{-i})) + \alpha_{\tau(i)} v_{\pi(\tau(i))} \geq \alpha_{\tau(i)} \frac{v_i}{2}$$

Now we take the sum over all players and take the expectation given the outcome v_i. Here the expectation is and over all realizations of other random variables keeping v_i fixed. We have
$$\mathbb{E}[\sum_i u_i(b'_i(v_i), b_{-i}(v_{-i}))] \geq \frac{1}{2}\mathbb{E}[\sum_i \alpha_{\tau(i)} v_i] - \mathbb{E}[\sum_i \alpha_{\tau(i)} v_{\pi(\tau(i)}]$$

Recognizing terms on the right hand side we see that $\sum_i \alpha_{\tau(i)} v_i = Opt$ and $\sum_i \alpha_{\tau(i)} v_{\pi(\tau(i))} = SW$. So we can put this as
$$\mathbb{E}[\sum_i u_i(b'_i(v_i), b_{-i}(v_{-i}))] \geq \frac{1}{2}\mathbb{E}[Opt] - \mathbb{E}[SW] \tag{1}$$

It is worth noting here that this inequality is very similar to the smoothness inequality that we have seen earlier, with $\lambda = \frac{1}{2}$ and $\mu = 1$. Now the final step, we use the fact that its a Bayes Nash Equilibrium and so any deviation is worse for any individual. So we have

$$\alpha_{\tau(i)} v_{\pi(i)} \geq \alpha_{\tau(i)}(v_{\pi(i)} - p_i) = u_i \geq u_i(b'_i(v_i), b_{-i}(v_{-i}))$$

Summing over all players we get

$$\sum_i \alpha_{\tau(i)} v_{\pi(i)} \geq \sum_i u_i(b'_i(v_i), b_{-i}(v_{-i}))$$

Taking the expectation and realizing $\sum_i \alpha_{\tau(i)} v_{\pi(\tau(i))} = SW$ we get

$$\mathbb{E}[SW] \geq \mathbb{E}[\sum_i u_i(b'_i(v_i), b_{-i}(v_{-i}))] \tag{2}$$

Combining the inequalities 1 and 2 gives us the desired result

$$\mathbb{E}[SW] \geq \frac{1}{4}\mathbb{E}[Opt]$$

1.4 Brief Sketch of PoA $\leq$ 3.16

Note that $3.16 = 2(1 - \frac{1}{e})^{-1}$ We will follow the same philosophy except that we will change the deviating bidding function. This time we choose the bidding function as $b'(v_i) \sim f_i(v_i)$ where $f_i(z)$ has support in $[0, (1 - \frac{1}{e})v_i]$ and

$$f_i(z) = \frac{1}{v_i - z}$$

and so in the case when player i gets the slot $\tau(i)$ we have

$$\mathbb{E}[u_i(b'_i(v_i), b_{-i}(v_{-i}))] = \mathbb{E}[\alpha_{\tau(i)}(v_i - b'_i(v_i))\mathbb{I}\{b'_i(v_i) \geq b_{\pi(\tau(i))}\}]$$

here $\mathbb{I}$ is the indicator function. The expectation is an integral

$$\int_0^{(1-\frac{1}{e})v_i} \alpha_{\tau(i)}(v_i - z)\mathbb{I}\{b'_i(v_i) \geq b_{\pi(\tau(i))}\}\frac{1}{v_i - z}dz$$

$(v_i - z)$ cancels away (it was chosen that way) and we are left with

$$\int_0^{(1-\frac{1}{e})v_i} \alpha_{\tau(i)}\mathbb{I}\{b'_i(v_i) \geq b_{\pi(\tau(i))}\}dz \geq \alpha_{\tau(i)}[(1 - \frac{1}{e})v_i - b_{\pi(\tau(i))}]$$

The inequality follows trivially from the definition of indicator function. We see that we get the same form of inequality as before except that instead of $\frac{1}{2}$ we now have $(1 - \frac{1}{e})$ and so the bound becomes

$$\mathbb{E}[SW] \geq \frac{1}{2}(1 - \frac{1}{e})\mathbb{E}[Opt] = \frac{1}{3.16}\mathbb{E}[Opt]$$