CS 6840 – Algorithmic Game Theory (3 pages) Feb. 6, 2012

Lecture 7: Generalized Utility Games

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1 Review: Facility Location Game

Recall the facility location problem discussed in last class. In this problem, there are a set of clients that need a service and a set of service providers. Each service provider i selects a location from possible locations A_i and offer price $p_{i,j}$ to client j. Note that providers may offer different prices to different clients.

Moreover, each location is associated with a cost $c_{i,j}$ for serving customer j from location i. We assume that client j has a value π_j for service. A strategy vector S is simply a vector of locations selected by each service provider. That is, $S = \{s_1, \dots, s_k\}$ for k providers.

We may consider this problem as a three-stage game as follows.

Stage 1. Providers select locations.

Stage 2. Providers setup prices for clients. Each provider i provides second cheapest cost among all other providers. That is $p_{i,j} = \min_{i \neq k} c_{k,j}$ by cheapest provider, and $c_{k,j}$ by everyone else.

Stage 3. Each user selects a provider for a service, and pay the specified price.

Last time, we also showed that this is a potential game with social welfare as a potential function:

$$\Phi = \sum_j (\pi_j - c_{i_j,j})$$

, where i_j is the min cost location to user j.

2 Generalized Frame for Utility Games

In this lecture, we would provide several desired properties on the utility games and derive useful results from these properties. Later, we will see that the facility location game is just one example of such games. The contents of this lecture came from the paper [Vetta2002] and Chapter 19 of the textbook.

Recall that in a utility game, each player i choose a location s_i. Social welfare is a function $U(S)$, where S is the vector of locations.

Here are the properties we require on the games:

Property 1. If player i selects location s_i and others select S_{-i} other locations. Player i gets utility $u_i(s_i, S_{-i})$. We assume

$$\sum_i u_i(s_i, S_{-i}) \leq U(S)$$

Property 2. $U(S) \geq 0$ and U is monotone on S. Also, $U(S)$ has decreasing marginal utility (same as submodular): for provider sets $X \subseteq Y$ and some extra service provider s, we have

$$U(X + s) - U(X) \geq U(Y + s) - U(Y)$$

Remark. While other requirements of this property are reasonable, the monotonicity requirement is questionable. This assumption ignores the cost of providing new service.

Lemma. The potential function of facility location game $\Phi = \sum_j (\pi_j - c_{i_j,j})$ has decreasing marginal utility property.

Proof. With X, more clients switch to s when added. Moreover, for each client j switching to s, its previous cost in $Y \leq$ cost in X. □

Property 3. $u_i(s_i, S_{-i}) \geq U(S) - U(S_{-i})$

Remark. Notice that in the facility location game, we have a stronger condition $u_i(s_i, S_{-i}) = U(S) - U(S_{-i})$.

3 Price of Anarchy

With the properties defined above, we next prove the main theorem of this lecture. We first recall the definition of (λ, μ)-smooth games.

Definition. A game is (λ, μ)-smooth if for all strategy vectors S, S^* we have

$$\sum_i u_i(s_i^*, S_{-i}) \geq \lambda \sum_i u_i(S^*) - \mu \sum_i u_i(S)$$

As shown in previous lectures, if the social welfare function is (λ, μ)-smooth, then we have the result that (social welfare at Nash) $\geq \frac{\lambda}{1+\mu}$ (optimal social welfare).

Next, we prove the main theorem of this class.

Theorem 1. Service location games are $(1, 1)$-smooth. (Hence, Nash $\geq \frac{1}{2}$ Optimal)

Proof. We first define $P_i^* = \{s_1^*, s_2^*, \ldots, s_i^*\}$, which is the first i prefix of S^*.

$$\begin{aligned}
\sum_i u_i(s_i^*, S_{-i}) &\geq \sum_i U(S_{-i} + s_i^*) - U(S_{-i}) && \text{(By Property 3)} \\
&\geq \sum_i U(S_{-i} + s_i^* + P_{i-1}^* + s_i) - U(S_{-i} + P_{i-1}^* + s_i) && \text{(By Property. 2)} \\
&= \sum_i U(S + P_i^*) - U(S + P_{i-1}^*) \\
&= U(S + S^*) - U(S) && \text{(telescoping sum)} \\
&\geq U(S^*) - U(S) && \text{(monotonicity)}
\end{aligned}$$

□

Remark. This property is not "strictly" (1,1)-smooth by definition, but very close. Note that the proof of Nash $\geq \frac{1}{2}$ Opt is not finished yet. The rest of proof will be shown in next class, using Property 1.

4 Extra Example

In this part, we show another example of utility games that satisfy the requirements.

In this example, there are still a set of clients and a set of service providers. We model life as a graph, as shown in Figure 1.

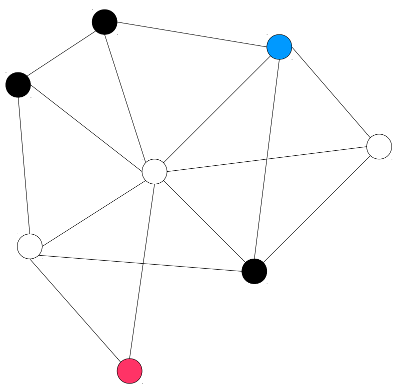

Figure 1: Example

In this graph, each node corresponds to a client. Service providers select among same nodes in the graph as their locations. For instance, service providers select the black nodes in Figure 1. There is an edge between a client and a location if the client is interested in the service on that node.

All users select providers in the following way:

- Each client selects the provider on the same node if there is one. For instance, the clients on black nodes select the providers on the same node.
- If there are providers on neighboring nodes, then client evenly share the services from those providers. For instance, the blue node shares the services provided by the two neighboring black nodes.

The unfortunate red node in Figure 1 gets no service since no neighbor is a provider.

The main difference between the facility location game and this example is that we assume the providers have to undercut other competitors' price in the previous one since only one provider could win the client; but that is not true in this example since services are shared among the competitors.

Due to the lack of time, we would not provide detailed utility function for this example. However, the point is that this example also satisfies all the required properties we used in this lecture.