CS 6840 – Algorithmic Game Theory (3 pages)	Spring 2012
Lecture 38 Notes	
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1 Lecture 38 – Friday 20 April 2012 - Price Equilibrium in Arrow-Debreu Model

1.1 Setup

- Goods $\{1, ..., k\}$.
- Players $\{1, ..., n\}$.
- Player i brings $\bar{w}_i = (w_1, ..., w_k)$ amount of goods to the market, and has utility $U_i(\bar{x}_i)$, where $\bar{x}_i = (x_{i1}, ..., x_{ik})$, where x_{ij} = amount of good j that player i gets.
- Assume utilities $U_i(\cdot)$ strictly monotone increasing, strictly concave, continuously differentiable.

1.2 Price Equilibrium

Let $p = (p_1, ..., p_k)$ be the prices for each good. Each player i sells $\bar{w}_i$ to get $p \cdot \bar{w}_i$ amount of money that is used for trading. Given prices, each player finds

$$\bar{x}_i = \arg\max_{\bar{x}} \{U_i(\bar{x}) : p \cdot \bar{x} \leq p \cdot \bar{w}_i, \bar{x} \geq 0\}$$

Note that since $U_i(\cdot)$ is strictly concave, $\bar{x}_i$ is unique. Also, since $U_i(\cdot)$ is strictly monotone increasing (in every dimension), $p \cdot \bar{x}_i = p \cdot \bar{w}_i$.

Definition. Prices $p = (p_1, ..., p_k), p_j > 0$ is a price equilibrium if the resulting $\bar{x}_1, ..., \bar{x}_n$ optima satisfy:

$$\forall j \qquad \sum_i x_{ij} \leq \sum_i w_{ij}$$

Note that by strict monotonicity of utilities, if $p_j = 0$ then all users want $x_{ij} = \infty$, so that cannot be an equilibrium.

Lemma (Market clearing)**.** For all goods j, $\sum_i x_{ij} = \sum_i w_{ij}$.

Proof. As noted earlier, we have

$$\begin{aligned} p \cdot \bar{x}_i &= p \cdot \bar{w}_i \\ \sum_i p \cdot \bar{x}_i &= \sum_i p \cdot \bar{w}_i \\ \sum_j p_j \sum_i x_{ij} &= \sum_j p_j \sum_i w_{ij} \end{aligned}$$

The only way for this to be equal is that they are term-by-term equal, so

$$\begin{aligned} p_j \sum_i x_{ij} &= p_j \sum_i w_{ij} \\ \sum_i x_{ij} &= \sum_i w_{ij} \end{aligned}$$

More generally, $p_j(\sum_i x_{ij}) - \sum_i w_{ij}) = 0$ even if $U_i(\cdot)$ is only monotone increasing.

Definition (Simplex)**.** $\Delta_n := \{x \in \mathbb{R}^n : x_i \geq 0, \sum_i x_i = 1\}$.

Theorem 1 (Brouwer Fixed Point Theorem)**.** If function $f : \Delta_n \to \Delta_n$ is continuous, then there exists x such that $f(x) = x$.

Theorem 2. Equilibrium prices exist.

Proof. Note that if p is a price equilibrium, then αp is also a price equilibrium for any $\alpha > 0$. WLOG, restrict to prices such that $p \in \Delta_n$. Let $\bar{x}_1, ..., \bar{x}_n$ be user optima, and let

$$\begin{aligned} e_j &= \left[\sum_j (x_{ij} - w_{ij})\right]^+ \\ f(p) &= \bar{p} \\ \forall j \qquad \bar{p}_j &= \frac{p_j + e_j}{\sum_i (p_i + e_i)} \end{aligned}$$

Lemma 3. p is price equilibrium $\iff f(p) = p$.

Proof. Clearly, p is price equilibrium $\implies f(p) = p$. Thus, we only need to show that if p is not a price equilibrium, then p is not a fixed point of f. Note that price changes unless e_j/p_j is fixed for all j. We claim that there exist a good j such that $\sum_i x_{ij} \leq \sum_i w_{ij}$. Recall,

$$\sum_j p_j \sum_i x_{ij} = \sum_j p_j \sum_i w_{ij}$$

Hence, it cannot be the case that $e_j > 0$ and for all goods j, $\sum_i x_{ij} > \sum_i w_{ij}$. Thus, if p is not a price equilibrium, then there is some good j such that $e_j > 0$ and hence, there must be some good that will have its price reduced under f, so p is not a fixed point of f.

Lemma 4. f is continuous.

Proof. $\bar{p}$ is continuous, and e_j is continous, so we only need $\bar{x}_i$ to be continous for all players i. Using a fact from continuous optimization, optimizer $\bar{x}_i$ (unique) is a continuous function of p, so f is continuous.

Lemma 5. $f : \Delta_n \to \Delta_n$ is a function. If prices are zero, then $x_{ij} = \infty$ and e_j is unbounded. Hence, we need $\bar{x}_i$'s to be bounded to make e_j's bounded. To do this, we modify the user optimization to include an extra condition.

$$\bar{x}_i = \arg\max_{\bar{x}} \left\{ U_i(\bar{x}) : p\bar{x} \leq p\bar{w}, \quad \forall j, x_j \geq 0, \quad \forall j, x_j \leq \sum_i w_{ij} + 1 \right\}$$

Note that the last condition cannot be tight at the fixed point as it violates price equilibirum conditions. Hence, this does not change the problem, but ensures that $\bar{x}_i$'s are bounded, and $f : \Delta_n \to \Delta_n$ is indeed a function.

Applying Brouwer's fixed point theorem to f shows that price equilibrium p exists.