Lecture 8 Scribe Notes

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1 Review: Generalized Frame for Utility Games

Recall from last time, we have a utility game where each player i chooses a location s_i. Social welfare is a function $U(S)$, where S is the vector of locations.

We require the following properties:

1. If player i selects location s_i and others select locations S_{-i}, then player i gets utility $u_i(s_i, S_{-i})$. We assume
$$\sum_i u_i(s_i, S_{-i}) \leq U(S)$$

2. $U(S) \geq 0$ and U is monotone on S. Also, $U(S)$ has decreasing marginal utility. For provider sets $X \subseteq Y$ and some other service provider s, we have
$$U(X + s) - U(X) \geq U(Y + s) - U(Y)$$

3. $u_i(s_i, S_{-i}) \geq U(S) - U(S_{-i})$

   Note that in the case where this holds with equality, then this is a potential game, with $\Phi = U$.

2 Price of Anarchy

Last time, we got far in showing that the Price of Anarchy for this game is at most 2. We had shown a (1,1)-smoothness-like property, namely, given any S, S^*,

$$\sum_i u_i(s_i^*, S_{-i}) \geq U(S^*) - U(S) \tag{1}$$

To complete the proof, we can take S^* to be an optimal solution, and S to be a Nash equilibrium solution. Since S is Nash, we also have the property

$$\sum_i u_i(s_i, S_{-i}) \leq U(S) \tag{2}$$

Combining these two equations, we have

$$\begin{aligned} U(S) &\geq U(S^*) - U(S) \\ 2U(S) &\geq U(S^*) \\ U(S) &\geq \frac{1}{2} U(S^*) \end{aligned}$$

This tells us that the utility of a Nash solution is at least half the utility of an optimal solution, giving us a Price of Anarchy of at most 2.

3 Finding a Good Quality Solution

There is the question of how to find a Nash equilibrium in this game. What would happen if players keep playing their best responses? Today, we will consider only a basic case, where one individual tries playing a best response at a time. We observe that every best response will result in an increase in the potential. Eventually, we will reach a Nash Equilibrium, when we can no longer increase the potential. However, this could potentially take a very long time - each player could go through several of his strategies before we reach an equilibrium.

Our goal now is to maximize the social welfare after some reasonable number of steps. The best we can hope for is something $\Theta(k)$ steps, where k is the number of players - we'd like that each player gets a chance to move at least once. However, we want every player to have a chance to move, not just one or two players - that is, we want the steps to be somewhat evenly distributed among the players. The best we can do is to have the expected number of steps per player to be equal, so we will choose a player to play his best response uniformly at random during each step.

We will take a utility game with $U(S)$ as the potential function. We wish to analyze the quality of the solution after at least t steps, if a random player plays his best response during each step. For now, we will let t be an arbitrary number of steps, and make a probability bound based on the inequality we derive. Note that we can't guarantee anything, since we are analyzing a random process. Instead, we will be analyzing $E[U(S)]$.

Note that we can't hope to get any better than $U(S) \geq \frac{1}{2}U(S^*)$, since this is where we'd end up if we start in a Nash Equilibrium. Therefore, we will try to show that after t steps, we will have (if t is large enough):

$$E[U(S)] \geq \left(\frac{1}{2} - \epsilon\right) U(S^*)$$

We will use equation (1), except this time, we will have S^* being the optimal solution and S being the solution at some step. We define β_i for each player i as

$$\beta_i = u_i(s_i^*, S_{-i}) - u_i(s_i, S_{-i})$$

We point out the intuition that is β_i is small, then we're close to a Nash equilibrium, and if β_i is large, then in expectation, we will make a big increase in our potential function.

If we let S' be the solution after making a single best response, we see that the expected increase is:

$$\begin{aligned} E[U(S') - U(S)] =& \frac{1}{k}\sum_i (\max_z u_i(z, S_{-i}) - u_i(s_i, S_{-i})) \\ \geq& \frac{1}{k}\sum_i (u_i(s_i^*, S_{-i}) - u_i(s_i, S_{-i})) \\ =& \frac{1}{k}\sum_i \beta_i \end{aligned}$$

where the first inequality follows on the basis that z is the best response for the player.

Using (1), we have

$$u_i(s_i^*, S_{-i}) \geq U(S^*) - U(S)$$

Also, by definition,

$$\sum_i u_i(s_i^*, S_{-i}) = \sum_i (u_i(s_i, S_{-i}) + \beta_i)$$

Combining these, we see

$$U(S) + \sum_i \beta_i \geq U(S^*) - U(S)$$
$$\sum_i \beta_i \geq U(S^*) - 2U(S)$$

and thus, we derive

$$E[U(S') - U(S)] \geq \frac{1}{k}(U(S^*) - 2U(S))$$

And hence, in expectation, $U(S)$ increases by at least $\frac{1}{k}(U(S^*) - 2U(S))$ after each step.

Now, we consider the expression

$$U(S^*) - 2U(S)$$

This decreases in expectation by $\frac{2}{k}(U(S^*) - 2U(S))$ in each step. If we let the initial value be $U(S^*) - 2U_0$, then after t steps,

$$\begin{aligned} U(S^*) - 2E[U(S)] &\leq \left(1 - \frac{2}{k}\right)^t (U(S^*) - 2U_0) \\ &= \left(1 - \frac{2}{k}\right)^t U(S^*) \end{aligned}$$

Rearranging this, we get $E[U(S)] \geq (\frac{1}{2} - \frac{1}{2}(1 - \frac{2}{k})^t)U(S^*)$. To obtain a convenient bound, we can choose $t = \frac{k}{2} \ln{(2\epsilon)^{-1}}$. Then, using the identity that $\lim_{x\to\infty}(1 - \frac{1}{x})^x = \frac{1}{e}$ from below, this is equivalent to $E[U(S)] \geq (\frac{1}{2} - \epsilon)U(S^*)$. Summarizing this result in theorem form, we have:

Theorem 1. After $\frac{k}{2} \ln{(2\epsilon)^{-1}}$ steps, we have that $E[U(S)] \geq \left(\frac{1}{2} - \epsilon\right) U(S^*)$.