Lecture 1 Scribe Notes

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Lecture 37 – Wednesday 18 April 2012 - Network resource games

Last time: network resource game on a single edge

Last time we talked about bandwidth sharing in a game with a single edge. In this setting there is a single edge with capacity B and each player i has a concave, continuously differentiable utility function $U_i(x)$. The players bids w_i for a unit of bandwidth and the mechanism gives each player $x_i = Bw_i/\sum_j w_j$ bandwidth. We showed that a Nash exists if there exists a p such that either $U_i'(x)(1 - \frac{x_i}{B}) = p$ or $U_i'(0) \leq p$ and $x_i = 0$. We showed that at Nash $p = \sum_j w_j/B$ and the price of anarchy is at most 4/3. We will use this price of anarchy bound in this lecture.

Last time: network resource game in a graph

We expanded the setting of the single edge to a graph $G = (V, E)$ where a game is played independently for each edge $e \in E$. Here each edge has a capacity b_e and each player has a path P_i that they would like to route through. Player i bids w_{ie} for all $e \in P_i$ and, if $w_{ie} = 0$, can additionally request a bandwidth amount x_{ie}. If $\sum_j w_{je} > 0$ then player i is allocated $x_{ie} = w_{ie}/\sum_j w_{je}$ bandwidth on edge e. Otherwise, if $\sum_j w_{je} = 0$, the players are allocated bandwidth according to their requests. If the requested amounts for edge e, $\sum_j x_{je} \leq b_e$, then the players are allocated their request; if $\sum_j x_{je} > b_e$ then all players are allocated 0.

The bandwidth player i can route along P_i is $x_i = \min_{e \in P_i} x_{ie}$ and player i receives utility $U_i(x_i)$. The bids are at Nash if $p_e = \sum_j w_{je}/b_e$ and either $x_i = 0$ and $U_i'(0) < \sum_{e \in P_i} p_e$ or $U_i'(0) = \sum_{e \in P_i} p_e(\frac{1}{1 - x_i/b_e})$.

Price of anarchy for network resource game in a graph

Theorem 1. The price of anarchy in the network resource game in a graph has price of anarchy at most 4/3.

Proof. Suppose that $x_1 \ldots x_n$ is a Nash. We will use the specific knowledge we have about $U_i(x)$ to do algebraic manipulations until the utility of player i decomposes according to the edges in P_i.

$$\begin{aligned} U_i(x) &\leq U_i(x_i) + (x - x_i)U_i'(x_i) \\ &= U_i(x_i) + (x - x_i)\sum_{e \in P_i} p_e \frac{1}{1 - x_i/b_e} \\ &\leq U_i(x_i) + \sum_{e \in P_i} (x_e - x_i) p_e \frac{1}{1 - x_i/b_e} \end{aligned}$$

The final inequality follows because the bandwidth player i can route along P_i (the quantity that is relevant for the utility function) is a min over the amount he can route over each edge.

The function described in the final inequality is a vectorized function $V_i(\bar{x})$ where $\bar{x} = (x_e)_{e \in P_i}$. We can consider the optimization problem $\max_{\bar{x}} \sum_i V_i(\bar{x})$. Because of the chain of inequalities above the optimal value of this optimization problem can only be larger than the optimal value of our original game.

Additionally, we claim that for any edge e, $x_1 \dots x_n$ serves as a Nash for the game on e, where $x_{ie} = x_i$ if $e \in P_i$ and $x_{ie} = 0$ otherwise. To see this we note that the utility player i receives is $U_i(x_i) + \sum_{e \in P_i} (x_e - x_i) p_e \frac{1}{1 - x_i/b_e}$ and that the derivative of this function at x_i with respect to x_e is $p_e \frac{1}{1 - x_i/b_e}$. Using the result on single edge games, the fact that this is the value of the derivative implies we are at Nash on e. Since this is true for all edges and players we are at Nash on every edge.

Combining the fact that the optimization problem using the vectorized utility function has larger optimal value, that $x_1 \dots x_n$ is a Nash on each edge, and the result that the price of anarchy is at most 4/3 on each edge we get the desired conclusion by a simple summation. □

Equilibria

In future lecture we will discuss the existence and computability of price equilibria and Nash equilibria in various settings. In general, the existence of an equilibria doesn't imply that they are easy to compute and we will exhibit some examples. So far, however, we have only considered cases where price equilibria are tractable to compute if we make use of the power of non-linear programming.

We can make the setting we have been in so far somewhat more general. Suppose there are k goods available and there is an amount m_j of each good j. There are n users and user i receives utility $U_i(\bar{x_i})$ for receiving $\bar{x_i} = (x_{i1} \dots x_{ik})$ of each good. The utility player i receives when paying price p is $U_i(\bar{x_i}) - p$. There are two things to note about this setting: the function $U_i(\bar{x_i})$ need not decompose by item and each player has an infinite amount of money to spend.

If we are to maximize utility we can write a non-linear program where all of the non-linearity is in the objective function:

$$\begin{aligned} \max \sum_i U_i(\bar{x_i}) & \\ \text{s.t.} \quad \bar{x_i} \geq 0 & \qquad \forall i \\ \sum_j x_{ij} \leq m_j & \qquad \forall j \end{aligned}$$

When the $U_i(\bar{x_i})$ are convex then this optimization problem can be solved easily via convex programming (modulo the expression of the convex function).

Definition. A function $f(x)$ is convex if $\frac{1}{2}f(x) + \frac{1}{2}f(x') \leq f(\frac{1}{2}x + \frac{1}{2}x')$ for all x and x' in the domain of f.

An additional byproduct of solving this optimization problem is that we can find prices $\bar{p} =$

$(p_1 \dots p_k) \geq 0$ that constitute a price equilibria. If $\bar{p}$ is a price equilibria we have that for all i

$$\begin{aligned} \bar{x}_i = \max_{\bar{x}} U_i(\bar{x}) - \bar{p}^T \bar{x} \\ \text{s.t} \qquad \sum_l x_{lj} \leq m_j \qquad \forall j \\ p_j > 0 \Rightarrow \sum_l x_{lj} = m_j \end{aligned}$$

We can use the tools of convex optimization to find such prices.

Fact 2. If $p_1 \dots p_k$ and $x_1 \dots x_n$ are a price equilibrium for the k products then $\sum_i U_i(x_i)$ is optimal.

Proof. Clearly $x_1 \dots x_n$ are feasible. Suppose $x_1^* \dots x_n^*$ are optimal. Then

$$\begin{aligned} U_i(x_i) - p^T x_i &\geq U_i(x_i^*) - p^T x_i^* \qquad \forall i \\ \sum_i U_i(x_i) - p^T x_i &\geq \sum_i U_i(x_i^*) - p^T x_i^* \\ \sum_i U_i(x_i) - U_i(x_i^*) &\geq \sum_i p^T x_i - p^T x_i^* \\ \sum_i U_i(x_i) - U_i(x_i^*) &\geq p^T \sum_i (x_i - x_i^*) \end{aligned}$$

The right hand side of the final inequality is always positive since at a price equilibrium when $p_j > 0$, $\sum_i x_{ij} = m_j \geq \sum_i x_{ij}^*$. This implies $\sum_i U_i(x_i) \geq \sum_i U_i(x_i^*)$ and optimality follows. □

The limitations of this method of using convex optimization are threefold. Firstly, we must have utilities that are convex functions; this excludes reasonable things like step functions for utilities. Secondly, the items up for bid must be continuously divisible; we can't add an integrality constraint on how much of an item a player receives. Finally, there is the unrealistic assumption that each player has an unbounded quantity of money to offer.