CS 6840 – Algorithmic Game Theory (3 pages)	Spring 2012
Lecture 2 Scribe Notes	
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1 Lecture 2 – Wednesday 25 January 2012 - Congestion Games

1.1 Definition

Definition. Congestion games are a class of games defined as follows:

- base set of congestable elements E
- n players
- each player i has finite set of strategies S_i
- a strategy $P \in S_i$ where $P \subseteq E$
- given a strategy P_i for each player i

$$x_e = \#\{i; e \in P_i\} \text{ for } e \in E$$

- player i choosing strategy P_i experiences delay

$$\sum_{e \in P_i} d_e(x_e)$$

Remark. Strategies P_i for player i define *Pure Nash Equilibrium* iff no one player can improve the delay by changing to another strategy Q_i.

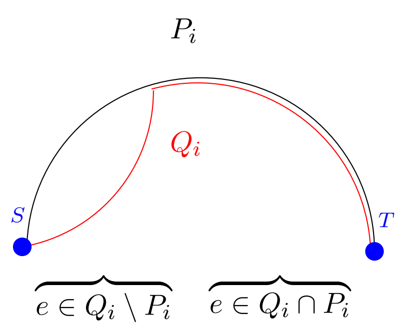

Figure 1: Strategies P_i, Q_i

Consider a player in the game shown in Figure 1. In this game, player i is switching from P_i to Q_i. As depicted, P_i and Q_i might have common parts as well as parts that differ. By switching, player i would experience the same delay in the parts that are common for P_i and Q_i and she will experience delay that results from adding one more person in Q_i in the parts that differ.

For all players i & all other $Q_i \in S_i$

$$\sum_{e \in P_i} d_e(x_e) \leq \sum_{e \in P_i \cap Q_i} d_e(x_e) + \sum_{e \in Q_i \setminus P_i} d_e(x_e + 1)$$

1.2 Equilibrium at Congestion Games

Now, lets explore the following questions:

- Does a general congestion game have a Nash Equilibrium?
- Are reasonable players able to find the Nash Equilibrium?

When we are looking for the Nash Equilibrium, the trivial approach is to change the strategy of one player and see if the resulting state is a Nash Equilibrium. In this approach, it is important to make sure that cycles do not occur to guarantee that the Nash Equilibrium is found. To see how cycles might occur, consider the following Matching Pennies Game.

Side Note If there are cycles present in the game, the equilibrium may not be found.

Example: Matching Pennies Game

- 2 players
- Strategies: H, T
- Rule:

$$m(s) = \begin{cases} \mathit{player}\ 1\ \mathit{wins} & \text{if strategies match} \\ \mathit{player}\ 2\ \mathit{wins} & \text{otherwise} \end{cases}$$

- Best response: Player starts with arbitrary strategy, switches if she loses

$$(H \mid H) \rightarrow (H \mid T) \rightarrow (T \mid T) \rightarrow (T \mid H) \rightarrow$$

1.3 Existence of Nash Equilibrium

Theorem 1. Congestion games repeated best response always finds the Nash Equilibrium.

Proof. Congestion games have a potential function Φ s.t. best response improves this function:

$$\Phi = \sum_{e} \sum_{k=1}^{x_e} d_e(k)$$

Consider: Player i switches from P_i to Q_i, change in Φ:

- edges $e \in P_i \setminus Q_i$ decrease by $d_e(x_e)$

- edges $e \in Q_i \setminus P_i$ increase by $d_e(x_e + 1)$

Note that, $\sum_{k=1}^{x_e} d_e(k)$ is the discrete integral of x_e, i.e. the potential function Φ is the summation of the discrete integral of x_e over all edges and the change in the potential function is equal to the change in a player's delay when she switches from strategy P_i to Q_i.

> When a player changes from strategy P_i to Q_i, the change in the delay is equal to the change in the potential function Φ.

Alternate Proof. Solution minimizing Φ is the Nash Equilibrium, assuming there are a finite number of solutions and a minimum exists. Since we assumed $d_e(x)$ is a monotone increasing (i.e. non-decreasing) function, there exists only one minimum, i.e. the Nash Equilibrium.