CS 6840 – Algorithmic Game Theory (3 pages) Spring 2012

Lecture 19 – A Game with Private Information

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Note: For administrative details regarding Homework 3 and the project, please view the first 7 minutes of the VideoNote lecture.

1 Introduction

In Lecture 17, we discussed two types of single-item auctions: first-price and second-price. For the former, we applied a Bayesian framework, in which we assumed that players independently draw values from a publicly known distribution, and use a single bidding function that is monotone in their values. Under this framework, we observed that the two auctions are:

- Outcome equivalent: the player with the highest value wins.
- Revenue equivalent: the payment collected from the winner is equal in expectation.

In general, two forms of auctions may not be outcome equivalent. However, even under more sophisticated Bayesian frameworks, outcome equivalence implies revenue equivalence (the revenue equivalence theorem). We consider a game under which the latter holds.

2 The Game

Suppose now that:

- Player i has a private value v_i drawn from distribution $\mathcal{F}_i$
- Player values are independent of another.
- Distributions $\mathcal{F}_i$ are public knowledge, but values are not.
- Players have individual bidding functions $b_i(v_i)$.

We construct the following mechanism, which converts bids into outcomes and payments:

1. Players submit bids $b_i(v_i)$.
2. Mechanism gives each player an amount $X_i \geq 0$ (possibly a random variable), and charges each player price p_i. Net value for each player is $v_i X_i - p_i$.

Remark: Letting $X_i \in \{0, 1\}, \sum_i X_i = 1, p_i = b_i(v_i)$ for the "high bidder" (and zero for everyone else), we recover a single-item auction. Letting $X_i \in [0, 1]$, we obtain a lottery where each player pays p_i for a probability X_i of "winning" the item.

3 Revenue Equivalence

The Nash equilibrium strategies for this game can be neatly characterized:

Theorem: The bid functions form a Nash equilibrium if and only if:

1. For each i,
$$x_i(v_i) = \mathbb{E}[X_i | v_i = v]$$
is nondecreasing in v

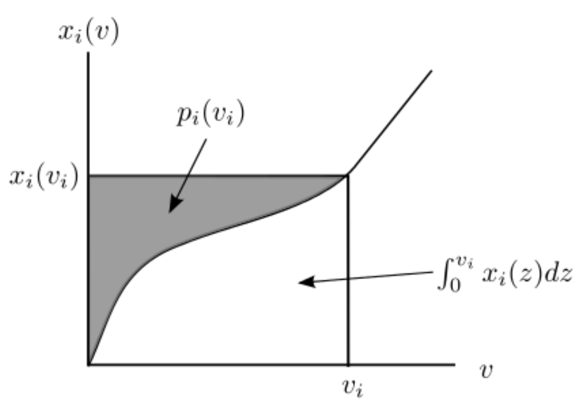
2. Prices $p_i(v_i) = \mathbb{E}[p_i | v_i = v]$ satisfy
$$p_i(v_i) = x_i(v_i)v_i - \int_0^{v_i} x_i(z)\,\mathrm{d}z + p_i(0)$$

Where the expectation is taken over other players' value distributions $\mathcal{F}_i$.

Remark: By statement 2, because player payments depend only on $x_i(v_i)$ (the outcomes), revenue equivalence follows by corollary (with the additional assumption that $p_i(0) = 0$).

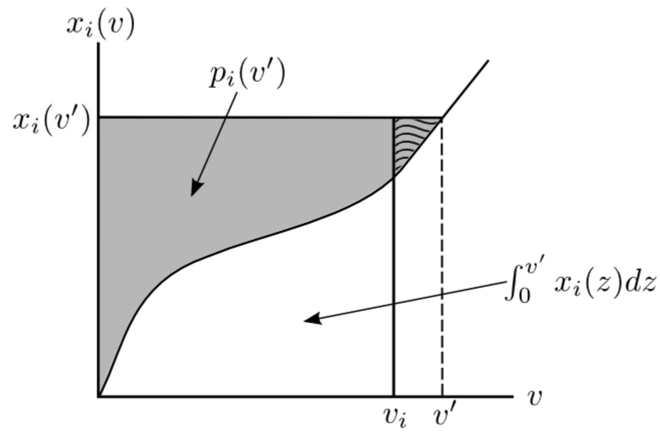
Proof: (NE $\implies$ 1) Suppose there exists a player i and values $v < v'$ such that $x_i(v) > x_i(v')$. By definition of Nash equilibrium, if player i's value is v, he prefers placing a bid using his true value to bluffing a value v':
$$x_i(v)v - p_i(v) \geq x_i(v')v - p_i(v')$$
Similarly, if player i has value v', he prefers not to bluff value v:
$$x_i(v')v' - p_i(v') \geq x_i(v)v' - p_i(v)$$
Summing the two equations, canceling, and regrouping terms, we get
$$\begin{aligned} x_i(v)v + x_i(v')v' &\geq x_i(v')v + x_i(v)' \\ [x_i(v) - x_i(v')]v &\geq [x_i(v) - x_i(v')]v' \\ [x_i(v) - x_i(v')](v - v') \geq 0 \end{aligned}$$
Since we assumed $v < v'$ and $x_i(v) > x_i(v')$, contradiction.

(1 & 2 $\implies$ NE) By picture. For convenience, assume that the bid functions are onto (the theorem still holds if this is relaxed). Consider the following plot of $x_i(v_i)$ versus v:



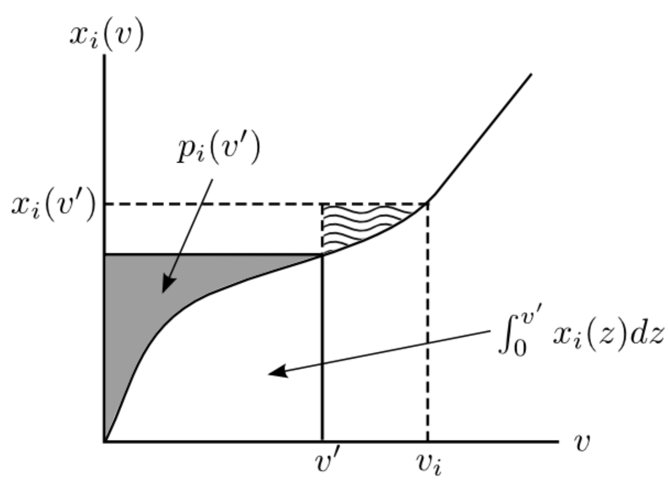
(Because we assumed that bid functions are onto, we can ignore jumps in the graph.) The area bounded by the rectangle represents the value of the item that player i receives. The shaded area is player i's payment, the white area his net payoff.

If player i bluffs a value $v' > v_i$, consider the following plot:



Although he increases the amount he receives to $x_i(v')$, player i values the item at $x_i(v')v_i$, the area of the solid rectangle. However, his payment increases to $p_i(v')$, resulting in a net loss represented by the wavy area in the graph.

If player i bluffs a value $v' < v_i$, a similar phenomenon occurs:



Player i thus could have increased his net value by the wavy area if he bid according to his true value v_i. This implies that player i has no incentive to place a bid different from that corresponding to his true value. We conclude that we are at a Nash equilibrium.

To be continued...