Revenue optimal auction with independent, irregular values

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1.1 Monotone Hazard Rate Assumption

Recall that, cumulative distribution function(cdf), $F(v) = \Pr(x \leq v)$, is a good way to describe distributions, but we prefer to think in its inverse, $v(q) = F^{-1}(1-q)$.

A distribution is regular if $q \cdot v(q)$ is concave, or equivalently, its derivative, $(q \cdot v(q))'$, is monotone non-increasing. Notice that this term is earlier defined as the virtual value, that is,

$$\phi(q) := (q \cdot v(q))' = v(q) + q \cdot v'(q)$$

Alternatively, this term could also be expressed in value space as $v - \frac{1-F(v)}{F'(v)} = v - \frac{1-F(v)}{f(v)}$.

$v(q)$ is monotone decreasing by definition. Thus, a sufficient condition for a distribution to be regular is that the second term of its derivative, $q \cdot v'(q)$, being monotone non-increasing. Or equivalently in value space, $\frac{1-F(v)}{f(v)}$ being monotone non-increasing.

Definition (Monotone Hazard Rate assumption)**.** We assume the hazard rate function, $\frac{f(v)}{1-F(v)}$ is monotone non-decreasing. This guarantees our distribution being regular.

Note that the hazard rate function, $\frac{f(v)}{1-F(v)}$, denotes the conditional probability that the event happens at price v, conditioned on event happens above v. (Or in other terms, the probability an item fails exactly at time v, given it survived until v.)

1.2 Non-regular Distribution

If we denote the revenue fixed at price $v(q)$ with probability q, the revenue generated is $R(q) = q \cdot v(q)$. When the distribution is non-regular, we had come up with the new function $\bar{R}(q)$ that is the minimum concavified function of $R(q)$ in the last lecture. In other words, $\bar{R}(q) \geq R(q)$, and $\bar{R}(q)$ is also the minimum function that is concave. Last time we showed $\bar{R}(q)$ is achievable. Today we want to show this is also the best we can achieve.

Lemma 1. $\bar{R}(q)$ is maximum revenue achievable.

Proof. We apply exactly the same notations as before, where:

$$\begin{aligned} x_i(v) &= E[x|v_i = v] \\ p_i(v) &= E[p_i|v_i = v] \\ \xi_i(q) &= E[x|v_i = v(q)] \\ \pi_i(q) &= E[p_i|v_i = v(q)] \end{aligned}$$

Recall that in Nash Equilibrium for any game, we have the following equalities:

1. x_i is monotone non-decreasing in v, or equivalently, $\xi(q)$ is monotone non-increasing in q.
2. $p_i(v) = p_i(0) + v \cdot x_i - \int_0^v x_i(z)dz$, or equivalently, $\pi_i(v) = \pi_i(1) - \int_q^1 v_i(r)\xi'(r)dr$.

We also showed that the price for person i in expectation is $E[\pi(q)] = \int_0^1 R'(q) \cdot \xi(q)dq$, where recall again, we defined the virtual value as $\phi(q) := R'(q)$.

Now to the proof, we have:

$$\int_0^1 \phi(q)\xi(q)dq = \int_0^1 R'(q)\xi(q)dq \tag{1}$$
$$= \int_0^1 \bar{R}'(q)\xi(q)dq + \int_0^1 (\bar{R}'(q) - R'(q))\xi(q)dq \tag{2}$$
$$= \int_0^1 \bar{R}'(q)\xi(q)dq + \int_0^1 [(R(q) - \bar{R}(q))\xi(q)]'dq - \int_0^1 (R(q) - \bar{R}(q))\xi'(q)dq \tag{3}$$

Step (3) follows with integration by parts for the second term in (2).

Now notice that since $R(0) = R(1) = \bar{R}(0) = \bar{R}(1) = 0$ by definition, the second term in (3) is 0. We also notice that $R(q) - \bar{R}(q) \leq 0$ by our construction, and $\xi'(q) \leq 0$ guaranteed by NE property (1), we know $(R(q) - \bar{R}'(q))\xi'(q) \geq 0$. Thus, the third term is less or equal to 0.

Then, from (3):

$$\int_0^1 \phi(q)\xi(q)dq \leq \int_0^1 \bar{R}'(q)\xi(q)dq \tag{4}$$

Which is the desired bound. □

1.3 Mechanism to Maximize Profit(Myerson's Optimal Mechanism)

Setup. We have players $1, 2, \ldots, n$, each with value $v_i \sim \mathcal{F}_i$.

Theorem 2. The feasible solution $\xi_i(q)$ that maximizes

$$\sum_i \int_0^1 \bar{R}_i'(q)\xi_i(q)dq$$

maximizes the profit.

Note that $\bar{R}_i'(q)$ is commonly denoted as ironed virtual value.

Sketch of proof. Due to the time constraints we only went through a rough sketch of the proof.

Firstly we notice that $\int_0^1 \bar{R}'(q)\xi(q)dq = \int_0^1 R'(q)\xi(q)dq$ iff. $\bar{R}(q) > R(q) \rightarrow \xi'(q) = 0$. (Easily shown by integration by parts) Since we showed that $\bar{R}(q)$ is the maximum achievable, guaranteeing $\bar{R}(q) > R(q) \rightarrow \xi'(q) = 0$ would yield maximum profit.

Then we need to prove that when we asked players for their values and ran this mechanism, one of the optimal allocation has the following guarantee: (Since certain $\bar{R}'(q)$ values could be held constant, there might potentially be infinitely many equivalent optimal allocations. We only need to show that there exists one satisfies these properties)

1. $\xi(q)$ is monotone decreasing in q

2. $\xi'(q) = 0$ for all $\bar{R}(q) > R(q)$

1. can be intuitively seen since we know $\bar{R}'(q)$ is monotone decreasing from concavity, and higher multiplier should give better allocation. This follows similar argument as used in VCG, and could be more rigorously proved with an exchange argument.

2. can be intuitively seen since in the interval (a, b) where $\bar{R}(q) > R(q)$, $\bar{R}'(q)$ is a fixed constant due to our concavification ($\bar{R}(q)$ being linear). When presented with this optimization problem, any allocation $\xi(q)$ over this range (a, b) is equivalent up to the objective function, as long as total sum of allocation into the range, $\int_a^b \xi(q)dq$ is held the same. (When multiplier is the same, assigning allocation to anywhere in the range would result in the same change in objective function) Thus, there exists one allocation where $\xi(q)$ is held constant in the range. In this case, $\xi'(q) = 0$.