CS 6840 – Algorithmic Game Theory (5 pages)	Spring 2012
Scribe Notes - Lecture 20	
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Lecture 20 – Monday March 3 2012 - Game with Private Information - continued

Please see sections 2.4 and 2.5 in Hartline's *Approximation and Economic Design* for a discussion similar to these notes.

1 Summary of lecture.

Nash equilibria - Review and discuss conditions for Nash equilibrium in Bayes payment/outcome games.

Revenue equivalence - Find that if 0-valued agents have zero cost, mechanisms are revenue equivalent, i.e. that if mechanisms produce the same outcomes in Nash equilibrium, they have the same expected revenue.

Importance of revenue equivalence - An auctioneer is free to choose outcomes, and thus can choose them to maximize revenue (this means that auctioneers just have to solve an optimization problem to optimize their revenue).

From value space to probability space - We will introduce a change of variables that takes us from value space to probability space. This change of variables will ultimately simplify the mathematics required to solve the above optimization problem, as well as provide valuable insight.

2 Review of Bayes payment/outcome game set up and conditions for Nash equilibrium.

Review of the game:

- player i has value v_i, where v_i is drawn independently from a distribution F_i
- player i's outcome is $X_i \geq 0$ and their payment is P_i
- utility for player i is $v_i X_i - P_i$. Observe that $v_i X_i$ is the value player i enjoys given the outcome of the game, and P_i is how much they have to pay for it.

Comment: we have been discussing the Nash equilibria of these games solely by thinking about outcomes rather than bidding structures.

IMPORTANT: There was a major error in the definitions of x_i and p_i from the previous lecture. The error is fixed in the posted lecture notes. Note that the proof of the erroneous theorem was actually a valid proof of the correct theorem. We will state the correct theorem here, and discuss why the proof from last time is applicable to this theorem and not the erroneous one.

2.1 Correcting the definitions from last lecture.

We define player i's expected outcome and expected payment as a function of their value v_i:

expected outcome - $x_i(v) = Exp(X_i | v_i = v)$

expected payment - $p_i(v) = Exp(P_i | v_i = v)$

The expectations are conditioned on player i's value v_i being equal to v and integrated over all possible values v_j taken from F for all players $j \neq i$. These statements encode the qualitative idea that this is a game with private information.

[The error from last lecture: we defined x_i and p_i to be conditioned on the OTHER players' values - this doesn't make sense, given that this is a private information game. Rather, it makes sense to **condition on our value**, and consider the **expectation of other players' values**.]

2.2 Theorem giving the conditions for Nash equilibrium.

Theorem 1. A game with outcomes and payments is in a Nash equilibrium iff it has the following properties:

monotonicity the expected outcome, $x_i(v_i)$, is monotone in v_i.

payment identity the expected payment is given by

$$p_i(v) = vx_i(v) + p_i(0) - \int_0^{x_i(v)} x_i(z)dz. \tag{1}$$

$p_i(0)$ is the required payment given you don't value the item - this term is typically zero, as one usually wants a game with free participation.

2.3 Discussion of proof from last lecture.

Monotonicity: Also see Hartline p 32. Let $v' \geq v$. If you are a player with value v and you consider bluffing and naming another value v', then you will experience the outcome

$$\underbrace{vx_i(v) - p_i(v)}_{\text{outcome if you name true value}} \geq \underbrace{vx_i(v') - p_i(v')}_{\text{outcome if you bluff}}. \tag{2}$$

If you are a player with value v' and you consider bluffing and naming another value v, then you will experience the outcome

$$\underbrace{v'x_i(v') - p_i(v')}_{\text{outcome if you name true value}} \geq \underbrace{v'x_i(v) - p_i(v)}_{\text{outcome if you bluff}}. \tag{3}$$

Adding these two inequalities we get

$$(v' - v)(x_i(v') - x_i(v)) \geq 0. \tag{4}$$

This implies that $v' \geq v$ implies $x_i(v') \geq x_i(v)$.

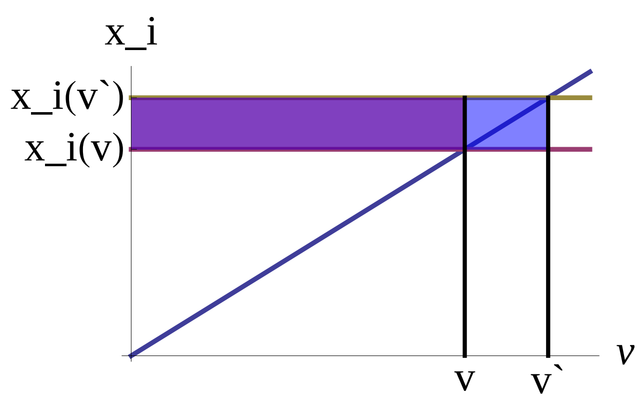
[Comment: This is true in the current game, but couldn't have been true last time. We were conditioning on other people's values: how could we bluff if we were talking about expectations of our value and other players' fixed values? It doesn't make sense to condition on other people's values, because we don't know them - this is a game with private information! We can only condition on our own value, and change our behavior with respect to that value. We can only consider the other players' expected values (i.e. behaviors).]

Payment identity: We can rearrange inequalities 2 and 3 to observe:

$$v(x_i(v') - x_i(v)) \leq p_i(v') - p_i(v) \leq v'(x_i(v') - x_i(v)). \tag{5}$$

This gives us lower and upper bounds on the price difference between v and v'.

Figure 1:



Say v' is a little larger than v, i.e. $v' = v + \delta$. Graphically this means the difference in $p(v') - p(v)$ must be at least the purple area in figure 1, and at most the blue + purple area. This means that as we increase v' the price goes as the blue area. This is a pictorial proof of the theorem's forward statement of the payment identity, that the price is

$$p(i) = \underbrace{vx_i(v)}_{\text{whole box}} - \underbrace{\int_0^v x_i(z)dz}_{\text{area under curve}} . \tag{6}$$

So far we have the forward direction of the theorem: if the game is in a Nash equilibrium, the outcomes and payments of players must be given by statements (1) and (2) in the theorem. Last time we give a proof of both directions. For our purposes today, it's satisfactory to focus only on the forward direction.

3 Revenue equivalence and its implications.

If we add the condition that $p_i(0) = 0$ for all i, then the payment identity shows **revenue equivalence**: the outcomes determine the payments.

Why do we care about revenue equivalence? Given you are an auctioneer who controls the outcomes, if you have revenue equivalence you also control the revenue! Thus, to maximize your revenue, you simply need to choose outcomes x_i that make p_i as high as possible: your task becomes a simple optimization problem!

3.1 How to use revenue equivalence in auction design.

To maximize the expectation, evaluate the expected v subject to the distribution F. This is, **find x_i that are monotone in v such that the expected payments are maximal**:

$$max \sum_i Exp_{F_i}(p_i(v)) \tag{7}$$

This problem could become awkward: we are maximizing a function that is a double integral over the distribution F_i. (The expectation involves an integral of $p_i(v)$ over F_i, and $p_i(v)$ is itself an integral over F_i.)

3.1.1 Simplify the integrals: change from value space to probability space.

Cumulative distribution maps values to probabilities - $F_i(v) = Pr(z \leq v) = 1 - q$ is the cumulative distribution, that is, it's the probability that we would sample a value less than v.

Inverse cumulative distribution maps probabilities to values - With probability q a player will have a value above $v = F^{-1}(1-q)$.

A one-to-one correspondence - F gives a one-to-one correspondence between q and v. Probability that you're less than some very small value is 0, probability that you're less than some very large value is 1. Probability that you're less than v is $1-q$. Probability you're greater than v is q.

The buying probability as an interpretation of q - q can be interpreted as the **buying probability**: a player will value the item above v with probability $q = Pr(v > z) = 1 - Pr(v \leq z) = 1 - F(v)$.

THE PLAN: Instead of sampling and integrating over v in $F(v)$, we will sample and integrate over $q \in [0,1]$. Sampling over q will be a nicer process to think about: v comes from F_i which is a 'weird' distribution, whereas q is a probability and thus comes from the 'friendly' interval $[0,1]$.

- $v_i(q)$ is the value that corresponds to probability q, i.e. $v_i(q) = F^{-1}(1-q)$.
- When we sampled a value v from F, we were thinking about the expected payments and outcomes given the price of the item is v: $x_i(v)$ and $p_i(v)$
- Now that we're sampling a probability q from [0,1] we will be thinking about the expected payments and outcomes given the item will be sold with probability q: $x_i(v(q)) = x_i(q)$ and $p_i(v(q)) = p_i(q)$.

Next time we will develop a theorem that gives the conditions for Bayes-Nash equilibrium in probability space.

Summary: sampling q instead of v turns awkward integrals into friendly integrals.

4 Thinking about a transaction with a single item and a single user.

How would you sell a single item to a single user if you're revenue maximizing?

Buyers - There is one buyer with distribution F_i.

Products - You have one item to sell.

What do you do? There is no competition between the buyers, your sole action is to set the price p. So, you ought to set p to maximize revenue.

Revenue curve expressed in terms of the price you set p - You will make $p * Pr(v > p) = p(1 - F(p))$. Expected revenue with price v: $v(1 - F(v))$.

Revenue curve expressed with probability of buying q - The probability a player will buy is q and the associated price is $v_i(q)$: you will make revenue $q * v_i(q)$.

Preview: Ultimately our goal will be to implement a change of variables from value to virtual value. This will both simplify the calculus and give insight into where the formula for revenue comes from.