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Lecture 23: Optimal Auctions for Non-Regular Distributions

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Please see section 3.4 in Hartline's *Approximation and Economic Design* for a discussion similar to these notes.

Review

We have an auction game in which the value v of each player is drawn from some distribution F, where $F(v) := Pr(X \leq v)$.

We use the function $v(q) := F^{-1}(1-q)$. This is the value such that $F(v(q)) = 1 - q$, or equivalently the value such that $Pr(X > v(q)) = q$.

Now, we look at an auction game with a single player and a single item. The value v of the player is drawn from some distribution F. If the reserve price of the item offered is $r = v(q)$, then the auctioneer's revenue is $R(q) = v(q)q$.

We make the following assumptions:

- F is continuous, differentiable, and invertible
- v is in $[0, v_{max}]$

The distribution F is called *regular* if R is a concave function. The distribution F is called *irregular* if R is a non-concave function. Now, let's consider an example with a non-regular distribution function.

Non-Regular Example

We have an auction game with a single player and single item, in which the value v of the player for the item is 1 or N.
The values are distributed such that:

- $Pr(v = 1) = 1 - \epsilon$
- $Pr(v = N) = \epsilon$

where $\epsilon < \frac{1}{N}$.

Now, how can we sell to this single person in the auction game? If the distribution was regular, we could run a second-price auction with reserve price r. There are two reasonable choices for r:

- $r = 1^-$, which gives a probability of sale 1 and revenue 1
- $r = N^-$, which gives a probability of sale ϵ and revenue $\epsilon \times N$

Since $\epsilon < \frac{1}{N}$, then $\epsilon N < 1$, and we can maximize revenue by running a second-price auction with reserve price 1^-. Yet, the distribution is not regular, and we claim there is something better to do.

Recall the following theorem:

Theorem 1. *Allocation and payment rules are in Bayes-Nash equilibrium if*

- (monotonicity) $x_i(q_i) = E[x_i(q_i)|v_i = v(q)]$ *is monotone non-increasing*
- (monotonicity) $x_i(v_i) = E[x_i(v)|v_i = v]$ *is monotone non-decreasing*
- (payment identity) $p_i(q_i) = -\int_{q_i}^{1} v_i(r)x_i'(r)dr + p_i(1)$

and the expected revenue from player i is $p_i(1) + \int_0^1 \phi_i(q)x_i(q)$, where $\phi_i(q) = R(q)'$.

Setting a single reserve price r corresponds to the following allocation function:

$$x_i(v_i) = \begin{cases} 0 & \text{if } v_i < r \\ 1 & \text{if } v \geq r \end{cases}$$

The theorem only specifies that $x_i(v_i)$ is a monotone function. Thus, the following monotone allocation function $x_i(v_i)$ is also valid:

$$x_i(v_i) = \begin{cases} 0 & \text{if } 0 < v < 1 \\ p & \text{if } 1 < v < N \\ 1 & \text{if } v > N \end{cases}$$

Informally, we do the following: we set two reserve prices. If you're willing to pay 1 for a good, then participate in a lottery. Otherwise, if you're willing to pay close to N, then you just get the good.

The corresponding expected payment is:

$$p_i(v_i) = \begin{cases} 0 & \text{if } 0 < v < 1 \\ p & \text{if } 1 < v < N \\ p + N(1-p) & \text{if } v > N \end{cases}$$

which is p with probability $1-\epsilon$ and $p + N(1-p)$ with probability ϵ for a total of $p(1-\epsilon) + \epsilon(p + N(1-p)) = p + \epsilon N(1-p)$ which is also at most 1 (since $\epsilon < \frac{1}{N}$).

However, this auction will do a lot better when there are n participants. The values of the participants are distributed as before. In this context, our selling scheme is as follows: If no player of value N shows up, select player at random and sell the item for price 1. Otherwise, give it to player of value N for price N. We are certain to get a price of 1, but if a high-valued player shows up, we will get higher revenue. This performs noticeably better than a fixed price auction.

Now, we look at the general case with an irregular distribution.

Ironed Revenue Curves

Let R be the revenue function. Let $\bar{R}$ be the smallest concave function that upper-bounds R. We make the following claim:

Lemma 1. *For any q, the maximum revenue possible with probability q of selling the good is $\bar{R}(q)$, and that revenue is achievable.*

Proof. Let $\hat{q}$ be arbitrary. We first show that the revenue $\bar{R}(\hat{q})$ is achievable. If $R(\hat{q}) = \bar{R}(\hat{q})$, then we sell with reserve price $v(\hat{q})$ to achieve the desired revenue.
Otherwise, $R(\hat{q}) < \bar{R}(\hat{q})$. Then, there must be two points a, b such that $a < \hat{q} < b$ and the line segment between points $(a, R(a)), (b, R(b))$ at $\hat{q}$ is

strictly above $R(\hat{q})$. Consider the following allocation rule, analagous to the allocation rule in the non-regular example:

$$x^{\hat{q}}(q) = \begin{cases} 1 & \text{if } q < a \\ \frac{\hat{q}-a}{b-a} & \text{if } q \in [a, b] \\ 0 & \text{if } b < q \end{cases}$$

Then, the probability a participant is allocated the item is $1 \times a + \frac{\hat{q}-a}{b-a} \times (b - a) = \hat{q}$. The revenue with this allocation rule is $R(a) + \frac{\hat{q}-a}{b-a}(R(b) - R(a)) = \bar{R}(\hat{q})$. Thus, the revenue $\bar{R}(\hat{q})$ is always achievable.
For the other direction see next class.

$\square$