CS 6840 – Algorithmic Game Theory (3 pages)	Spring 2012
Lecture 4 Scribe Notes	
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1 Price of anarchy in non-atomic congestion games

The price of anarchy is a measure of the quality of Nash solutions compared to a centrally designed optimal solution. We consider non-atomic congestion games with:

- congestible elements E
- user types $i = 1, \dots, n$
- strategy sets S_i for all i
- congestion $d_e(x)$ along element e given x users
- f_p users choosing strategy p

Define the congestion along each element e as

$$x_e = \sum_{p|e \in p} f_p,$$

and recall that equilibrium is attained if for all $f_p > 0$, user types i, and $p, q \in S_i$ we have that $\sum_{e \in p} d_e(x_e) \leq \sum_{e \in q} d_e(x_e)$; alternatively, equilibrium is attained when

$$\phi = \sum_e \int_0^{x_e} d_e(\xi)\, d\xi$$

is minimized.

1.1 Measuring the quality of solutions

- Sum of delays / average delay
- Maximum delay
- Pareto optimal – doesn't require a shared objective.

Note that minimizing average delay implies Pareto optimality. We consider minimizing average delay, or minimizing

$$\sum_p f_p \sum_{e \in p} d_e(x_e) = \sum_e d_e(x_e) \sum_{p \in e} f_p = \sum_e d_e(x_e) x_e.$$

Definition. Delay is (λ, μ)-smooth if for all $x, y > 0$

$$y d(x) \leq \lambda y d(y) + \mu x d(x).$$

We choose x as a Nash solution and y as an optimal solution.

Lemma. The linear delay function $d(x) = ax + b$ is $(1, 1/4)$-smooth for $a, b \geq 0$.

Proof. We want to show that $y(ax+b) \leq y(ay+b)+\frac{1}{4}x(ax+b)$. Let $x, y \geq 0$ be given, and suppose first that $x \leq y$. Then

$$y(ax+b) \leq y(ay+b)$$
$$y(ax+b) \leq y(ay+b) + \frac{1}{4}x(ax+b)$$

since each term is non-negative. Now consider the case when $y < x$. We want to show that $yd(x) \leq yd(y) + \frac{1}{4}xd(x)$, or

$$yd(x) - yd(y) \leq \frac{1}{4}xd(x)$$
$$y(ax+b) - y(ay+b) \leq \frac{1}{4}x(ax+b)$$
$$ayx - ay^2 \leq \frac{1}{4}ax^2 + \frac{1}{4}xb.$$

Since $b \geq 0$ it is sufficient to show that

$$ayx - ay^2 \leq \frac{1}{4}ax^2.$$

If $a = 0$, we are done. If $a > 0$, we are interested in upper-bounding $ayx - ay^2$ with respect to y. Using elementary calculus we can see that the function $f(y) = axy - ay^2$ attains a maximum value when $y = x/2$, and so we have that

$$ayx - ay^2 \leq ax \cdot \frac{x}{2} - a\frac{x^2}{4} = \frac{1}{4}ax^2,$$

as required. ☐

Theorem 1. Suppose the delay function is (λ, μ)-smooth. If a flow f is a Nash equilibrium and a flow f^* is optimal (with respect to the sum of delays) then

$$\sum_e x_e d_e(x_e) \leq \frac{\lambda}{1-\mu} \sum_e x_e^* d_e(x_e^*).$$

Proof. Let p_j and p_j^* be paths between the same source and sink at Nash equilibrium and optimality, respectively, and let δ_j flow along p_j at Nash and along p_j^* at optimality. Since p_j is at Nash, we have that

$$\sum_{e \in p_j} d_e(x_e) \leq \sum_{e \in p_j^*} d_e(x_e)$$
$$\sum_j \delta_j \sum_{e \in p_j} d_e(x_e) \leq \sum_j \delta_j \sum_{e \in p_j^*} d_e(x_e)$$
$$\sum_e d_e(x_e) \sum_{p_j | e \in p_j} \delta_j \leq \sum_e d_e(x_e) \sum_{p_j^* | e \in p_j^*} \delta_j$$
$$\sum_e d_e(x_e) x_e \leq \sum_e d_e(x_e) x_e^*.$$

Since d_e is (λ, μ)-smooth, we have

$$\begin{aligned}
\sum_e d_e(x_e)x_e &\leq \sum_e d_e(x_e)x_e^* \\
\sum_e d_e(x_e)x_e &\leq \lambda \sum_e x_e^* d_e(x_e^*) + \mu \sum_e x_e d_e(x_e) \\
\sum_e d_e(x_e)x_e - \mu \sum_e x_e d_e(x_e) &\leq \lambda \sum_e x_e^* d_e(x_e^*) \\
(1-\mu) \sum_e d_e(x_e)x_e &\leq \lambda \sum_e x_e^* d_e(x_e^*) \\
\sum_e d_e(x_e)x_e &\leq \frac{\lambda}{1-\mu} \sum_e x_e^* d_e(x_e^*),
\end{aligned}$$

as required. □