Algorithmic Game Theory	**Problem Set 1**
CS 684 Spring 2008	**Due Wednesday, February 17th**

We will maintain a FAQ for the problem set on the course Web page. You may use any fact we proved in class without proving the proof or reference, and may read the relevant chapters of the book. However, you may not use other published papers, or the Web to find your answer.

A full solution for each problem includes proving that your answer is correct. If you cannot solve a problem, write down how far you got, and why are you stuck. You may solve the problem with a partner. Please hand in a shared problem set with both your names on the solutions. We are happy to help finding partners, let us (Renato or Hu) know if you don't have a partner, and would like one.

Solutions can be submitted on CMS, or handed in at my office. Please type your solution, to make it easier to read.

(1) Solve problem 17.2 from the book.

(2) During the first class we considered a resource allocation game where n players are sharing bandwidth of 1. (Example 1.4 in the book). Each player i chooses an amount $x_i \geq 0$ and the utility of player i is $U_i(x_i, x_{-i}) = x_i(1 - \sum_j x_j)$. Note that this utility is a bit different from the one in class, if $\sum_j x_j > 1$ we will have all players with negative utility. We showed that the game has a unique Nash equilibrium. Recall that a game is a potential game if there is a function $\Phi(x)$ such that for any state x of the game, any player i, and any alternate strategy x'_i for this player $\Phi(x'_i, x_{-i}) - \Phi(x_i, x_{-i}) = U_i(x'_i, x_{-i}) - U_i(x_i, x_{-i})$. Is the bandwidth sharing game a potential game?

(3) Consider the load balancing game: there are n jobs, each controlled by a separate and selfish user. There are m servers S that can serve jobs, and each job j has an associated set $S_j \subseteq S$ of servers where it can possibly be served. For this problem we assume that the load of each jobs is 1, and each server i has a load dependent response time: $r_i(x)$ is the response time of server i if its load is x. This is a congestion game, and hence Nash equilibria are local optima of the corresponding potential function Φ. This problem explores of we you can find a socially optimal solution, or a Nash in polynomial time. **Hint:** You may use the fact that the minimum cost matching problem (defined below) can be solved in polynomial time. This may be useful as a subroutine.

We assume that $r_i(x)$ is a monotone increasing function for all i. You may also assume that $r_i(x)$ is convex is that helps. Please note in your answer what assumption you are using and where (r monotone or convex?).

- **(a)** Give a polynomial time algorithm to find an equilibrium.
- **(b)** We consider two possible definitions of social optimum for this game. First consider the assignment of jobs to servers that minimizes the maximum response time, and give a polynomial time algorithm to find the best assignment for this objective function.
- **(c)** Next consider the assignment of jobs to servers that minimizes social welfare, the sum of all response times **over jobs** (or average response time), and give a polynomial time algorithm to find the best assignment for this objective function.

The minimum cost matching problem is given by a bipartite graph G, costs on the edges and an integer k, and the problem is to find a matching in G of size k of minimum possible cost.

(4) Consider the following grouping game. Players are nodes of a graph, and edges represent social relations. We assumed that edge pair of nodes i and j has one associated with the pair $u_{ij} \geq 0$, where $u_{ij} \geq 0$ is the benefit players i and j both incur if nodes i and j are together. In the game each player can choose between affiliating with one of two social groups A or B. The result is a partition of the nodes into two sets. The utility of a player $i \in A$ is $U_i(A, B) = \sum_{j \in A} u_{ij}$, and if player i is in B its utility is $U_i(A, B) = \sum_{k \in B} u_{ik}$, i.e., the some of the utilities for partners in the same group. Assume that the values are symmetric, namely that $u_{ij} = u_{ji}$ for all i and j.

(a) Show that this is a potential game with the social welfare $\frac{1}{2} \sum_i U_i(A, B)$ is a potential function.

(b) Show that the Price of Anarchy is at most 2 (i.e., that any (pure strategy) Nash equilibrium has social welfare at least 1/2 of the maximum possible.

(c) It turns out that all potential games are congestion games with the right definition of congestion (where the strategies of players are general sets, not paths in graphs). Define a congestion game that is equivalent to this partition game.

(d) Do either of the first two statements remain true without the assuming symmetry? that is if u_{ij} may not equal u_{ji}.

(5) Consider the non-atomic multicommodity flow problem with continuous monotone increasing delay functions $\ell_e(x)$ on the edges. We defined Nash equilibria as a flow f (a vector with coordinates f_P for paths P), such that for all pairs of paths P and Q connecting the same pair of terminals, if $f_P > 0$ then $\ell_P(f) \leq \ell_Q(f)$, where $\ell_P(f)$ is the delay with flow f.

A maybe more intuitive definition would be as follows. A flow f is a Nash equilibrium, if the following holds. For any pairs of paths P and Q connecting the same pair of terminals, such that $f_P > 0$ and any $0 < \delta \leq f_P$ if we define an alternate flow $\hat{f}$ defined below $\ell_P(f) \leq \ell_Q(\hat{f})$:

$$\hat{f}_R = \begin{cases} f_P - \delta & \text{if } R = P \\ f_Q + \delta & \text{if } R = Q \\ f_R & \text{otherwise} \end{cases}$$

This definition considers the flow f and a very small δ amount of flow on a path P, and wonders if this small amount of flow is happier on path P or should it switch to another path Q. Here we model the flow being non-atomic by allowing arbitrary small amounts of flow to switch, but we do not allow "zero" amount to switch, as it is less clear what that means.

Show that the two definitions are the same.

(6) Consider the non-atomic routing model, let $\ell_e(x)$ be the delay function of edge e. We have seen (Corollary 18.10) that when users evaluate delay via the the cost $\ell_e^*(x) = \ell_e(x) + x\ell_e'(x)$ than the resulting equilibrium flow is the socially optimal flow for the original delays ℓ. This cost consists of a selfish part $\ell_e(x)$ and an altruistic part $x\ell_e'(x)$ (the additional cost caused to others). Here we consider purely altruistic users, that evaluate delay via the $x\ell_e'(x)$, the purely altruistic part. Let f^a be such a purely altruistic equilibrium, where all flow is carried on paths that is shortest with respect to the cost $x\ell_e'(x)$.

(a) Prove that under the usual assumption that ℓ_e is monotone increasing and differentiable, such a purely altruistic equilibrium flow always exists.

(b) Assuming the delays are linear (i.,e., $\ell_e(x) = a_e x + b_e$ for all edges), can you say anything about the quality of such purely altruistic flows compared to the social optimum?