Algorithmic Game Theory	**Problem Set 2**
CS 684 Spring 2008	**Due Thursday, March 13, 2008**

We will maintain a FAQ for the problem set on the course Web page. You may use any fact we proved in class without proving the proof or reference. However, you may not use other published papers, or the Web to find your answer.

(1) Consider the load balancing game: there are n jobs, each controlled by a separate and selfish user. There are m servers S that can serve jobs, and each job j has an associated set $S_j \subseteq S$ of servers where it can possibly be served. For this problem we assume that the load of each jobs is 1, and each server i has a load dependent response time: $r_i(x)$ is the response time of server i if its load is x. We assume that $r_i(x)$ is a monotone increasing function for all i. You may also assume that $r_i(x)$ is convex is that helps. We showed in class that this is a potential game, that is the Nash equilibria of this game are the local optima for the objective function Φ used on Tuesday, January 29th. **Hint:** You may use the fact that the minimum cost matching problem (defined below) can be solved in polynomial time. This may be useful as a subroutine.

(a) Give a polynomial time algorithm to find an equilibrium.

(b) We consider two possible definitions of social optimum for this game. First consider the assignment of jobs to servers that minimizes the maximum response time, and give a polynomial time algorithm to find the best assignment for this objective function.

(c) Next consider the assignment of jobs to servers that minimizes the sum of all response times **over jobs** (or average response time), and give a polynomial time algorithm to find the best assignment for this objective function.

The minimum cost matching problem is given by a bipartite graph G, costs on the edges and an integer k, and the problem is to find a matching in G of size k of minimum possible cost.

(2) In class on February 7th we proved the bicriteria bound comparing a Nash flow to an optimal flow. When we want to consider capacitated edges, an edge with capacity u_e can be modeled by a delay function $\ell_e(x) = a_e/(u_e - x)$, where a_e is another constants associated with the edge e (in addition to the capacity u_e. (Note that this delay does model capacity u_e as the delay grows to infinity as the flow approaches the capacity.)

Show that the total delay of a Nash flow in such a network is bounded above by the minimum possible total delay of a flow satisfying the same demands in a network with only 1/2 the capacity, i.e., where each capacity is halved (u_e is replaced by $u_e/2$ for each e without changing a_e).

(3) In the bicriteria bound we proved on February 7th we assumed that all flow is very sensitive to delay, and the flow is in a real Nash equilibrium. It is maybe more reasonable to model a stable state as one of the approximate equilibria, i.e., assume only that for all pairs of paths P and Q connecting the same pair of terminals, if $f_P > 0$ then $\ell_P(f) \leq (1+\varepsilon)\ell_Q(f)$ for some sensitivity parameter $\varepsilon > 0$. Show that there is a version of the bicriteria bound that holds also for all approximate Nash flows.

(4) Show that in the fair-sharing mechanism, Nash equilibria must always allocate non-zero amount of the resource to at least two users.

(5) Extend the fair-sharing mechanism to a pair of resources, say with capacity C_1 and C_2 and 3 users. Users 1 and 2 wants only resource 1 and 2 respectively, and have a utility function $U_i(x)$ for the resource, while user 3 wants the same amount of both. So if allocated amounts y_1 and y_2 his/her utility is $U_3(\min(y_1, y_2))$. Assume that U_i is continously differentiable, bf strictly monotone increasing and concave. There are two variants of this game.

> *Users 1 and 2 offer amount w_i of money for the resource they want, user 3 offers amounts w'_1 and w'_2 for the two resources separately. Each resource is then allocated using fair sharing: $x_1 = C_1 \frac{w_1}{w_1+w'_1}$ for user 1, and $y_1 = C_1 \frac{w'_1}{w_1+w'_1}$, and similarly for resource 2.*

(a.) This allocation is very easy to implement, but it possibly allocates different amounts to user 3 of the two resources, which is not good for user 3. Show that this will not happen in equilibrium, that is $y_1 = y_2$ in any equilibrium.

(b.) Give the conditions of equilibrium for this game, and argue that equilibrium must exists.

Even though user 3 will get the same amount of the two resources at equilibrium, one may view this game as awkward, as it can allocate different amounts on the two edge to user 2. An alternate version would be the following.

> *Users 1 and 2 offer amount w_i of money for the resource they want, user 3 offers amounts w_3 for the two resources combined. We then solve the equation system $w'_1 + w'_2 = w_3$ and $C_1 \frac{w'_1}{w_1+w'_1} = C_2 \frac{w'_2}{w_2+w'_2}$. Then use w'_1 and w'_2 in the fair sharing mechanism, as above.*

Note that this system always allocates the same amount of both resources to user 3, but its harder to compute the allocation.

(c.) **Assume that $C_1 = C_2 = C$ and $U_1 = U_2$ for this part.** Give the conditions of equilibrium for this new game, and argue that equilibrium must exists. (Note that without some extra assumption the equilibrium does not always exists.)

(d.) This part is attempting to explore which of the two rules would the players prefer. Give an intuitive explanation of your finding. Consider the **further** special case when the utility of all players is $U_i(x) = x$. What are the optimal allocation, and the Nash allocations in the two different games. Which player prefers which of the rules?

(6) Consider the local network formation game of Section 19.2 (from February 26). Recall that in this game players are nodes, and each player chooses a subset of its neighbors to build an edge to it. The cost of α for every edge built, plus a cost derived by the pairwise distances in the resulting undirected graph.

(a.) Show that there is an n_0 so that a path of length $n \geq n_0$ is not an equilibrium in this game for any value of α. (Note that n_0 is a number, may not depend on α.)

An alternate variant of this game commonly used in the literature is when the edge between a pair of nodes u, v) has to be paid by both players (charging $\alpha/2$ to each). In this case, we will need two players to add any edge, so we consider a variant of Nash, when two player coalitions can coordinate a move together (player v and w can together drop any edges they currently pay for, and possibly buy the edge connecting them, but only if this move benefits **both** v and w). We will call such a solution "pairwise stable"

(b.) Show that for every $n > 0$, there is an $\alpha > 0$, so that the path of length n is a pairwise stable solution for the game.